MAPPING SPACES AND RIGIDIFICATION OF QUASI-CATEGORIES

DANIEL DUGGER AND DAVID SPIVAK

ABSTRACT. We provide some new models for the mapping spaces of a quasi-category, and a new construction for rigidifying a quasi-category into a simplicial category. These constructions come from the use of necklaces, which are simplicial sets obtained by stringing simplices together. As an application of these methods, we use our models to reprove some basic facts from [L] about the relation between quasi-categories and simplicial categories.

CONTENTS

1. INTRODUCTION

Quasi-categories are a certain generalization of categories, in which one has not only 1-morphisms but n-morphisms for every natural number n. They have been extensively studied by Joyal [J1], [J2] and by Lurie [L]. If K is a quasi-category and x and y are two objects of K, then one may associate a "mapping space" $K(x, y)$ which is a simplicial set. Mapping spaces are important for understanding quasi-categories; for instance, in Lurie's approach an equivalence of quasi-categories is defined to be a functor which is essentially surjective and which induces weak equivalences on all mapping spaces.

The trouble is that there are many different ways to exhibit mapping spaces, the different models being weakly equivalent but not isomorphic. No particular model is ideal for every application, and so one must become versatile at changing back-and-forth. In [L] several models are used, but there is one particular model denoted $\mathfrak{C}(K)(x, y)$ which admits composition maps $\mathfrak{C}(K)(y, z) \times \mathfrak{C}(K)(x, y) \to \mathfrak{C}(K)(x, z)$, giving rise to a simplicial category. The construction $\mathfrak{C}(K)$ may be thought of as a *rigidification* of the quasi-category K, and it is proven in [L] that the homotopy theories of quasi-categories and simplicial categories are equivalent via this functor.

In this paper we do two things. The main thing is that we introduce some new models for the mapping spaces $K(x, y)$, which are particularly easy to describe and particularly easy to use—in fact they are just nerves of ordinary categories (i.e., 1-categories). Our models also admit composition maps giving rise to a simplicial category, and so we are giving a new method for rigidifying quasi-categories. We prove that our construction is homotopy equivalent (as simplicial categories) to Lurie's $\mathfrak{C}(K)$. The second thing we do is explain how mapping spaces of quasi-categories fit into the well-understood theory of homotopy function complexes in model categories [DK3]. The latter technology immediately gives various tools for understanding why different models for these mapping spaces are weakly equivalent. By relating the homotopy function complex approach to our models via nerves of categories, we are able to give new proofs of some results about the rigidification process.

1.1. **Mapping spaces via simplicial categories.** Now we describe our results in more detail. Recall the Joyal model structure on the category $s\mathcal{S}et$: the cofibrations are the monomorphisms, the weak equivalences are the so-called *weak categorical equivalences* (see Section 2.3), and the fibrations are maps with the right lifting property with respect to acyclic cofibrations. We will denote this model structure by $s\mathcal{S}et_J$. Quasi-categories are defined to be the fibrant objects of $s\mathcal{S}et_J$, and they have a particularly nice description: a simplicial set is a quasi-category if it satisfies the right lifting property with respect to inner horn inclusions $\Lambda^n_i \to \Delta^n, \quad 0 < i < n$.

There is a functor, constructed in [L], which sends any simplicial set K to a corresponding simplicial category $\mathfrak{C}(K) \in s\mathcal{C}at$. This is the left adjoint in a Quillen pair

$$\mathfrak{C} \colon s\mathcal{S}et_J \rightleftarrows s\mathcal{C}at \colon N,$$

where N is called the **coherent nerve**. The functor N can be described quite explicitly (see Section 2), but the functor $\mathfrak{C}$ is in comparison a little mysterious. Each $\mathfrak{C}(K)$ is defined as a certain colimit in the category $s\mathcal{C}at$, but colimits in $s\mathcal{C}at$ are notoriously difficult to understand.

Our main goal in this paper is to give a different model for the functor $\mathfrak{C}$. Define a **necklace** (which we picture as "unfastened") to be a simplicial set of the form

$$\Delta^{n_0} \vee \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$$

where each $n_i \geq 0$ and where in each wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$. The simplices Δ^{n_i} where $n_i \geq 1$ are called the **beads** of the necklace.

The first and last vertex in any necklace T are denoted α_T and ω_T, respectively (or just α and ω if T is obvious from context). If S and T are two necklaces, then by $S \vee T$ we mean the necklace obtained in the evident way, by gluing the final vertex ω_S of S to the initial vertex α_T of T. Write $\mathcal{N}ec$ for the category whose objects are necklaces and where a morphism is a map of simplicial sets which preserves the initial and final vertices.

Let $S \in s\mathcal{S}et$ and let $a, b \in S_0$. If T is a necklace, we use the notation

$$T \to S_{a,b}$$

to indicate a morphism of simplicial sets $T \to S$ which sends α_T to a and ω_T to b. Let $(\mathcal{N}ec \downarrow S)_{a,b}$ denote the evident category whose objects are pairs $[T, T \to S_{a,b}]$ where T is a necklace. Note that for $a, b, c \in S$, there is a functor

$$(\mathcal{N}ec \downarrow S)_{b,c} \times (\mathcal{N}ec \downarrow S)_{a,b} \longrightarrow (\mathcal{N}ec \downarrow S)_{a,c}$$

which sends the pair $[T_2, T_2 \to S_{b,c}] \times [T_1, T_1 \to S_{a,b}]$ to $[T_1 \vee T_2, T_1 \vee T_2 \to S_{a,c}]$.

Let $\mathfrak{C}^{nec}(S)$ be the function which assigns to any $a, b \in S_0$ the simplicial set $\mathfrak{C}^{nec}(S)(a,b) = N(\mathcal{N}ec \downarrow S)_{a,b}$ (the usual nerve of the category $(\mathcal{N}ec \downarrow S)_{a,b}$). The above pairings of categories induces pairings on the nerves, which makes $\mathfrak{C}^{nec}(S)$ into a simplicial category with object set S_0.

Theorem 1.2. *There is a natural weak equivalence of simplicial categories $\mathfrak{C}^{nec}(S) \simeq \mathfrak{C}(S)$, for all simplicial sets S.*

In the above result, the weak equivalences for simplicial categories are the so-called "DK-equivalences" used by Bergner in [B]. See Section 2 for this notion.

In this paper we also give an explicit description of the mapping spaces in the simplicial category $\mathfrak{C}(S)$. A rough statement is given below, but see Section 4 for more details.

Theorem 1.3. *Let S be a simplicial set and let $a, b \in S$. Then the mapping space $X = \mathfrak{C}(S)(a,b)$ is the simplicial set whose n-simplices are triples subject to a certain equivalence relations. The triples consist of a necklace T, a map $T \to S_{a,b}$, and a flag $\overrightarrow{T} = \{T^0 \subseteq \cdots \subseteq T^n\}$ of vertices in T. For the equivalence relation, see Corollary 4.4. The face maps and degeneracy maps are obtained by removing or repeating elements T^i in the flag.*
The pairing

$$\mathfrak{C}(S)(b,c) \times \mathfrak{C}(S)(a,b) \longrightarrow \mathfrak{C}(S)(a,c)$$

sends the pair of n-simplices $([T \to S; \overrightarrow{T^i}], [U \to S, \overrightarrow{U^i}])$ to $[U \vee T \to S, \overrightarrow{U^i \cup T^i}]$.

Theorem 1.2 turns out to be very useful in the study of the functor $\mathfrak{C}$. There are many tools in classical homotopy theory for understanding the homotopy types of nerves of 1-categories, and via Theorem 1.2 these tools can be applied to understand mapping spaces in $\mathfrak{C}(S)$. We demonstrate this technique in Section 6 (and later in Section 9) by proving, in a new way, some of the basic properties of $\mathfrak{C}$ found in [L].

1.4. **Mapping spaces via model category theory.** The above theorems give two weakly equivalent models for the mapping spaces in a quasi-category. The perspective of homotopy function complexes, in the sense of Dwyer-Kan, leads to a collection of different models. In any model category $\mathcal{M}$, given two objects X and Y there is a homotopy function complex $\mathrm{hMap}_{\mathcal{M}}(X,Y)$. This can be defined in several ways, all giving weakly equivalent models:

(1) It is the mapping space between X and Y in the simplicial localization $L_W\mathcal{M}$, where one inverts the subcategory W of weak equivalences [DK1].
(2) It is the mapping space in the hammock localization $L_H\mathcal{M}$ constructed by Dwyer and Kan in [DK2].
(3) It can be obtained as the simplicial set $[n] \mapsto \mathcal{M}(Q^nX, \widehat{Y})$ where $Y \to \widehat{Y}$ is a fibrant replacement in $\mathcal{M}$ and $Q^\bullet X \to X$ is a cosimplicial resolution of X [DK3].
(4) It can dually be obtained as the simplicial set $[n] \mapsto \mathcal{M}(\tilde{X}, R_nY)$ where $\tilde{X} \to X$ is a cofibrant replacement and $Y \to R_\bullet Y$ is a simplicial resolution of Y.

(5) It can be obtained as the nerve of various categories of zig-zags, for instance the category whose objects are zig-zags
$$X \xleftarrow{\sim} A \longrightarrow Y$$
and where the maps are commutative diagrams equal to the identity on X and Y.

In some sense the models in (3) and (4) are the most computable, but one finds that all the models are useful in various situations. One learns, as a part of model category theory, how to compare these different models and to see that they are weakly equivalent. See [DK3] and [D1], as well as Section 7 of the present paper.

The above technology can be applied to the Joyal model structure in the following way. The overcategory $(\partial\Delta^1 \downarrow s\mathcal{S}et_J)$ inherits a model category structure from $s\mathcal{S}et_J$ ([H, 7.6.5]). Given a simplicial set K with chosen vertices a and b, consider K as a simplicial set under $\partial\Delta^1$ via the evident map $\partial\Delta^1 \to K$ sending $0 \mapsto a$, $1 \mapsto b$. In particular, we can apply this to Δ^1 and the vertices 0 and 1. This allows us to consider the homotopy function complex
$$\mathrm{hMap}_{(\partial\Delta^1 \downarrow s\mathcal{S}et_J)}(\Delta^1, K),$$
which somehow feels like the pedagogically 'correct' interpretation of the mapping space in K from a to b. Note, however, that it is "inert" in the sense that there is no composition law for mapping spaces of this type (see Remark **??**).

The following result is more like an observation than a proposition:

Proposition 1.5. *Let K be a quasi-category. The mapping spaces $\mathrm{Hom}^R_K(a,b)$ and $\mathrm{Hom}^L_K(a,b)$ of* [L] *are models for the above homotopy function complex, obtained via two different cosimplicial resolutions of Δ^1. The pullback*
$$\begin{tikzcd} K_{a,b} \arrow[r, dashed] \arrow[d, dashed] & K^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r, "(a,b)"] & K \times K \end{tikzcd}$$
is also a model for the same homotopy function complex, this time obtained via a third cosimplicial resolution, namely the one sending $[n]$ to the pushout
$$\Delta^1 \times \Delta^n \leftarrow \partial\Delta^1 \times \Delta^n \to \partial\Delta^1.$$

Note that, given the above proposition, the Dwyer-Kan technology shows immediately that the three constructions $\mathrm{Hom}^R_K(a,b)$, $\mathrm{Hom}^L_K(a,b)$, and $K_{a,b}$ are all weakly equivalent, and in fact gives a 'homotopically canonical' weak equivalence between any two.

1.6. **Connections between the two approaches.** What is not immediately clear is how to connect the homotopy function complex
$$\mathrm{hMap}_{(\partial\Delta^1 \downarrow s\mathcal{S}et_J)}(\Delta^1, K)$$
to the simplicial sets $\mathfrak{C}(K)(a,b)$ and $\mathfrak{C}^{nec}(K)(a,b)$. This is explained in Section 9, where they are shown to be connected by a canonical zig-zag of weak equivalences. At one level the connection can be seen as follows. For any necklace T, there is a canonical inclusion $T \hookrightarrow \Delta[T]$, where $\Delta[T]$ is the simplex with the same vertex

set as T; this inclusion is a weak equivalence in $s\mathcal{S}et_J$. Any map $T \to K$ therefore gives rise to a zig-zag

$$\Delta^1 \longrightarrow \Delta[T] \xleftarrow{\sim} T \to K$$

where the map $\Delta^1 \to \Delta[T]$ is the unique 1-simplex connecting the initial and final vertices. Zig-zags of the above type are known to give a model for homotopy function complexes.

It should be noted, though, that we were not able to find an argument for the result following this intuitive outline. Instead, we give a crafty argument using a variation on $\mathfrak{C}^{nec}$ in which one replaces necklaces by a more general class of "gadgets". See Section 9 for details.

1.7. **Relation with the work of Lurie.** In this paper we take as a given the Joyal model structure on $s\mathcal{S}et$, and from there we develop the properties of mapping spaces and categorification. In Lurie's book [L] he takes a different approach, where he starts by developing the properties of mapping spaces and categorification and then proves the existence of the Joyal model structure as a consequence of this work. His methods involve a detailed and lengthy study of what he calls "straightening and unstraightening" functors, and it was a vague disatisfaction with this material—together with the hope of avoiding it—that first led us to the work in the present paper.

Due to the inherent differences in the two approaches, it is slightly awkward for us to quote results from [L] without creating confusions and possible circularities. Because of this, there are a few minor results whose proofs we end up repeating or redoing in a slightly different way. The result is that the present paper can be read independently of [L]—although this should not be taken as a denial of the intellectual debt we owe to that work.

1.8. **Notation and Terminology.** We will sometimes use $s\mathcal{S}et_K$ to refer to the usual model structure on simplicial sets, which we'll term the *Kan model structure*. The fibrations are the Kan fibrations, the weak equivalences (called Kan equivalences from now on) are the maps which induce homotopy equivalences on geometric realizations, and the cofibrations are the monomorphisms.

We will often be working with the category $s\mathcal{S}et_{*,*} = (\partial\Delta^1 \downarrow s\mathcal{S}et)$. When we consider it as a model category, the model structure is imported from the Joyal model structure on $s\mathcal{S}et$; in this case we will denote it $(s\mathcal{S}et_{*,*})_J = (\partial\Delta^1 \downarrow s\mathcal{S}et_J)$. Note that $\mathcal{N}ec$ is a full subcategory of $s\mathcal{S}et_{*,*}$.

An object of $s\mathcal{S}et_{*,*}$ is a simplicial set X with two distinguished points a and b. We sometimes (but not always) write $X_{a,b}$ for X, to remind us that things are taking place in $s\mathcal{S}et_{*,*}$ instead of $s\mathcal{S}et$.

2. Background on quasi-categories

In this section we give the background on quasi-categories and simplicial categories needed in the rest of the paper.

2.1. **Simplicial categories.** A simplicial category is a category enriched over simplicial sets; it can also be thought of as a simplicial object of $\mathcal{C}at$ in which the categories in each level have the same object set. We use $s\mathcal{C}at$ to denote the category of simplicial categories. A cofibrantly-generated model structure on $s\mathcal{C}at$ was developed in [B]. A map of simplicial categories $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence (sometimes called a *DK-equivalence*) if

(1) For all $a, b \in \operatorname{ob}\mathcal{C}$, the map $\mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ is a Kan equivalence of simplicial sets;
(2) The induced functor of ordinary categories $\pi_0 F\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is surjective on isomorphism classes.

Likewise, the map F is a fibration if

(1) For all $a, b \in \operatorname{ob}\mathcal{C}$, the map $\mathcal{C}(a,b) \to \mathcal{D}(Fa, Fb)$ is a Kan fibration of simplicial sets;
(2) For all $a \in \operatorname{ob}\mathcal{C}$ and $b \in \operatorname{ob}\mathcal{D}$, if $e\colon Fa \to b$ is a map in $\mathcal{D}$ which becomes an isomorphism in $\pi_0\mathcal{D}$, then there is an object $b' \in \mathcal{C}$ and a map $e'\colon a \to b'$ such that $F(e') = e$ and e' becomes an isomorphism in $\pi_0\mathcal{C}$.

The cofibrations are the maps which have the left lifting property with respect to the acyclic fibrations.

Remark 2.2. The second part of the fibration condition seems a little awkward at first. In this paper we will actually have no need to think about fibrations of simplicial categories, but have included the definition for completeness.

Bergner writes down sets of generating cofibrations and acyclic cofibrations in [B].

2.3. **Quasi-categories.** A quasi-category is defined to be a simplicial set which has the right lifting property with respect to the inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for integers $0 < k < n$. It turns out that there is a unique model structure on $s\mathcal{S}et$ with the properties that

(i) The cofibrations are the monomorphisms;
(ii) The fibrant objects are the quasi-categories;

It is easy to see that there is at most one such structure. To do this, let E^1 be the 0-coskeleton—see [AM], for instance—of the set $\{0, 1\}$ (note that the geometric realization of E^1 is essentially the standard model for S^∞). The map $E^1 \to *$ has the right lifting property with respect to all monomorphisms (see Lemma 8.2), and so it will be an acyclic fibration in this structure. Therefore $X \times E^1 \to X$ is also an acyclic fibration for any X, and hence $X \times E^1$ will be a cylinder object for X. Since every object is cofibrant, a map $A \to B$ will be a weak equivalence if and only if it induces bijections $[B, Z] \to [A, Z]$ for every quasicategory Z, where $[A, Z]$ means the coequalizer of $s\mathcal{S}et(A \times E^1, Z) \rightrightarrows s\mathcal{S}et(A, Z)$. Therefore the weak equivalences are determined by properties (i)–(ii), and since the cofibrations and weak equivalences are determined so are the fibrations.

The existence of such a stucture is less clear, but was estalished by Joyal (see [J1] or [J2], or [L] for another proof). For this reason, we will call it the **Joyal model structure** and denote it by $s\mathcal{S}et_J$. Joyal [J2] calls the weak equivalences in this structure *weak categorical equivalences*, whereas in [L] they are just called categorical equivalences. In the present paper we will call them **Joyal equivalences**.

Remark 2.4. Before knowing that there exists such a model structure, let us consider the above remarks as simply a suggestion of terminology for maps in $s\mathcal{S}et$. To reiterate, a map $A \to B$ inducing bijections $[B, Z] \to [A, Z]$ for every quasi-category Z will be called a *Joyal equivalence*; a quasi-category will be called a *Joyal fibrant object*; a monomorphism will be called a *Joyal cofibration*.

In this paper we rarely need to know anything about the Joyal model structure, other than this this terminology, until Section 6. On occasion we will use the following facts about the structure that were proven by Joyal:

(1) If K and X are simplicial sets and X is fibrant in $s\mathcal{S}et_J$, then so is X^K ([J2, 4.5]). It would be great to prove this ourselves and thus improve this remark.
(2) The inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ (for $0 < k < n$) are Joyal equivalences.
(3) $s\mathcal{S}et_J$ is left proper.

The second fact is actually an easy consequence of the first, using $K = E^1$ and Property (ii) above; we leave the reader to puzzle this out. The third property is true simply because every simplicial set is Joyal cofibrant.

2.5. **Background on $\mathfrak{C}$ and N.** Given a simplicial category S, one can construct a fibrant object in $s\mathcal{S}et_J$ called the *coherent nerve* of S [L, 1.1.5]. We will now describe this construction.

Recall the adjoint functors $F\colon \mathcal{G}rph \rightleftarrows \mathcal{C}at\colon U$. Here $\mathcal{C}at$ is the category of 1-categories, and $\mathcal{G}rph$ is the category of graphs: a graph consists of an object set and morphism sets, but no composition law. The functor U is a forgetful functor, and F is a free functor. Given any category $\mathcal{C}$ we may then consider the comonad resolution $(FU)_\bullet(\mathcal{C})$ given by $[n] \mapsto (FU)^{n+1}(\mathcal{C})$. This is a simplicial object in the category $\mathcal{C}at$ of categories, and since all face and degeneracy maps are isomorphisms on object sets, $(FU)_\bullet(\mathcal{C})$ is a category enriched in simplicial sets; i.e. it is a simplicial category in the usual sense.

There is a functor of simplicial categories $(FU)_\bullet(\mathcal{C}) \to \mathcal{C}$ (where the latter is considered a discrete simplicial category). This functor induces a weak equivalence on all mapping spaces, a fact which can be seen by applying U, at which point the comonad resolution picks up a contracting homotopy. Note that this means that the simplicial mapping spaces in $(FU)_\bullet(\mathcal{C})$ are all homotopy discrete.

Recall that $[n]$ denotes the category $0 \to 1 \to \cdots \to n$, where there is a unique map from i to j whenever $i \le j$. We let $\mathfrak{C}(\Delta^n)$ denote the simplicial category $(FU)_\bullet([n])$. The mapping spaces in this simplicial category can be analyzed completely, and are as follows. For each i and j, let $P_{i,j}$ denote the poset of all subsets of $\{i, i+1, \ldots, j\}$ containing i and j (ordered by inclusion). Note that the nerve of $P_{i,j}$ is isomorphic to the cube $(\Delta^1)^{j-i-1}$ if $j > i$, Δ^0 if $j = i$, and the emptyset if $j < i$. The nerves of the $P_{i,j}$'s naturally form the mapping spaces of a simplicial category with object set $\{0, 1, \ldots, n\}$, using the pairings $P_{j,k} \times P_{i,j} \to P_{i,k}$ given by union of sets.

Lemma 2.6. *There is an isomorphism of simplicial categories $\mathfrak{C}(\Delta^n) \cong NP$.*

Remark 2.7. The proof of the above lemma is a bit of an aside from the main thrust of the paper, so it is given in Appendix A. In fact we could have *defined* $\mathfrak{C}(\Delta^n)$ to be NP, which is what Lurie does in [L], and avoided the lemma entirely; the construction $(FU)_\bullet([n])$ will never again be used in this paper. Nevertheless, the identification of NP with $(FU)_\bullet([n])$ seems informative to us, which is why we have included it.

For any simplicial category $\mathcal{D}$, the **coherent nerve** of $\mathcal{D}$ is the simplicial set $N\mathcal{D}$ given by

$$[n] \mapsto s\mathcal{C}at(\mathfrak{C}(\Delta^n), \mathcal{D}).$$

It was proven by Lurie [L] that $N\mathcal{D}$ is always a quasi-category; see also Lemma 6.5 below.

The functor N has a left adjoint denoted $\mathfrak{C}\colon s\mathcal{S}et \to s\mathcal{C}at$. Any simplicial set K may be written as a colimit of simplices via the formula

$$K \cong \operatorname*{colim}_{\Delta^n \to K} \Delta^n,$$

and consequently one has

$$\mathfrak{C}(K) \cong \operatorname*{colim}_{\Delta^n \to K} \mathfrak{C}(\Delta^n) \tag{2.7}$$

where the colimit takes place in $s\mathcal{C}at$. This formula is a bit unwieldy, however, in the sense that it does not give much concrete information about the mapping spaces in $\mathfrak{C}(K)$. The point of the next three sections is to obtain such concrete information, via the use of necklaces.

3. Necklaces

A necklace is a simplicial set obtained by stringing simplices together in succession. In this section we establish some basic facts about them, as well as facts about the more general category of ordered simplicial sets. When T is a necklace we are able to give a complete description of the mapping spaces in $\mathfrak{C}(T)$ as nerves of certain posets, generalizing what was said for $\mathfrak{C}(\Delta^n)$ in the last section. See Proposition 3.8.

As briefly discussed in the introduction, a **necklace** is defined to be a simplicial set of the form

$$\Delta^{n_0} \vee \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$$

where each $n_i \geq 0$ and where in each wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$. We say that the necklace is in **preferred form** if either $k = 0$ or each $n_i \geq 1$.

Let $T = \Delta^{n_0} \vee \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ be in preferred form. Each Δ^{n_i} is called a **bead** of the necklace. A **joint** of the necklace is either an initial or a final vertex in some bead. Thus, every necklace has at least one vertex, one bead, and one joint; Δ^0 is not a bead in any necklace except in the necklace Δ^0 itself.

Given a necklace T, write V_T and J_T for the sets of vertices and joints of T. Note that $V_T = T_0$ and $J_T \subseteq V_T$. Both V_T and J_T are totally ordered, by saying $a \leq b$ if there is a directed path in T from a to b. The initial and final vertices of T are denoted α_T and ω_T (and we sometimes drop the subscript); note that $\alpha_T, \omega_T \in J_T$.

Every necklace T comes with a particular map $\partial\Delta^1 \to T$ which sends 0 to the initial vertex of the necklace, and 1 to the final vertex. If S and T are two necklaces, then by $S \vee T$ we mean the necklace obtained in the evident way, by gluing the final vertex of S to the initial vertex of T. Let $\mathcal{N}ec$ denote the full subcategory of $s\mathcal{S}et_{*,*} = (\partial\Delta^1 \downarrow s\mathcal{S}et)$ whose objects are necklaces $\partial\Delta^1 \to T$. We sometimes talk about $\mathcal{N}ec$ as though it is a subcategory of $s\mathcal{S}et$.

A simplex is a necklace with one bead. A **spine** is a necklace in which every bead is a Δ^1. Every necklace T has an associated simplex and spine, which we now define. Let $\Delta[T]$ be the simplex whose vertex set is the same as the (ordered) vertex set of T. Likewise, let $\mathrm{Spi}[T]$ be the longest subnecklace of T that is a spine. Note that there are inclusions $\mathrm{Spi}[T] \hookrightarrow T \hookrightarrow \Delta[T]$. The assignment $T \to \Delta[T]$ is a functor, but $T \to \mathrm{Spi}[T]$ is not (for instance, the unique map of necklaces $\Delta^1 \to \Delta^2$ does not induce a map on spines).

Lemma 3.1. *For any necklace T, the maps $\mathrm{Spi}[T] \hookrightarrow T \hookrightarrow \Delta[T]$ are both weak equivalences in $s\mathcal{S}et_J$.*

Proof. It is easy to see that the map $\Delta^n \vee \Delta^1 \to \Delta^{n+1}$ is a composite of $\sum_{i=1}^{n} \binom{n}{i}$ cobase changes along inner horn inclusions

$$\Lambda_n^{\{a_1,\ldots,a_i,n,n+1\}} \longrightarrow \Delta^{i+1},$$

where $0 \leq a_1 < \cdots < a_i < n$. Consequently it is a Joyal equivalence. By induction, it follows that $\mathrm{Spi}[T] \to \Delta[T]$ is a Joyal equivalence. That $\mathrm{Spi}[T] \to T$ is a Joyal equivalence is proven, bead by bead, using cobase changes of the form $\mathrm{Spi}(B) \to B$. The result follows by two-out-of-three.

□

3.2. **Ordered simplicial sets.** If $T \to T'$ is a map of necklaces, then the image of T is also a necklace. To prove this, as well as for several other reasons scattered thoughout the paper, it turns out to be very convenient to work in somewhat greater generality.

If X is a simplicial set, define a relation on its 0-simplices by saying that $x \preceq y$ if there exists a spine T and a map $T \to X$ sending $\alpha \mapsto x$ and $\omega \mapsto y$. In other words, $x \preceq y$ if there is a directed path from x to y inside of X. Note that this relation is clearly reflexive and transitive, but not necessarily antisymmetric: that is, if $x \preceq y$ and $y \preceq x$ it need not be true that $x = y$.

Definition 3.3. *A simplicial set X is **ordered** if*

(i) The relation $\preceq$ defined on X_0 is antisymmetric, and

(ii) An simplex $x \in X_n$ is determined by its sequence of vertices $x(0) \preceq \cdots \preceq x(n)$; i.e. no two distinct n-simplices have identical vertex sequences.

Note the role of degenerate simplices in condition (ii). For example, notice that $\Delta^1/\partial\Delta^1$ is not an ordered simplicial set.

The following notion is also useful:

Definition 3.4. *Let A and X be simplicial sets. A map $A \to X$ is called a **simple inclusion** if it has the right lifting property with respect to the canonical inclusions $\partial\Delta^1 \hookrightarrow T$ for all necklaces T. (Note that such a map really is an inclusion, because it has the lifting property for $\partial\Delta^1 \to \Delta^0$).*

The notion of simple inclusion says that if there is a "path" (in the sense of a necklace) in X that starts and ends in A, then it must lie entirely in A. As an example, four out of the five inclusions $\Delta^1 \hookrightarrow \Delta^1 \times \Delta^1$ are simple inclusions.

Lemma 3.5. *A simple inclusion $A \hookrightarrow X$ has the right lifting property with respect to the maps $\partial\Delta^k \hookrightarrow \Delta^k$ for all $k \geq 1$.*

Proof. Suppose given a square

$$\begin{tikzcd} \partial\Delta^k \arrow[r] \arrow[d] & A \arrow[d] \\ \Delta^k \arrow[r] & X. \end{tikzcd}$$

By restricting the map $\partial\Delta^k \to A$ to $\partial\Delta^1 \hookrightarrow \partial\Delta^k$ (given by the initial and final vertices of $\partial\Delta^k$), we get a corresponding lifting square with $\partial\Delta^1 \hookrightarrow \Delta^k$. Since $A \to X$ is a simple inclusion, this new square has a lift $l\colon \Delta^k \to A$. It is not immediately clear that l restricted to $\partial\Delta^k$ equals our original map, but the two maps are equal after composing with $A \to X$; since $A \to X$ is a monomorphism, the two maps are themselves equal. $\square$

Lemma 3.6. *Let X and Y denote ordered simplicial sets and let $f\colon X \to Y$ be a map.*

(1) The category of ordered simplicial sets is closed under taking finite limits.
(2) Every necklace is an ordered simplicial set.
(3) If $X' \subseteq X$ is a simplicial subset, then X' is also ordered.
(4) The map f is completely determined by the map $f_0\colon X_0 \to Y_0$ on vertices.
(5) If f_0 is injective then so is f.
(6) The image of an n-simplex $x\colon \Delta^n \to X$ is of the form $\Delta^k \hookrightarrow X$ for some $k \leq n$.
(7) If T is a necklace and $y\colon T \to X$ is a map, then its image is a necklace.
(8) Suppose that $X \leftarrow A \to Y$ is a diagram of ordered simplicial sets, and both $A \to X$ and $A \to Y$ are simple inclusions. Then the pushout $B = X \amalg_A Y$ is an ordered simplicial set, and the inclusions $X \hookrightarrow B$ and $Y \hookrightarrow B$ are both simple.

Proof. For (1), the terminal object is a point with its unique ordering. Given a diagram of the form

$$X \longrightarrow Z \longleftarrow Y,$$

let $A = X \times_Z Y$. It is clear that if $(x, y) \preceq_A (x', y')$ then both $x \preceq_X x'$ and $y \preceq_Y y'$ hold, and so antisymmetry of $\preceq_A$ follows from that of $\preceq_X$ and $\preceq_Y$. Condition (ii) from Definition 3.3 is easy to check.

Parts (2)–(5) are easy, and left to the reader.

For (6), the sequence $x(0), \ldots, x(n) \in X_0$ may have duplicates; let $d\colon \Delta^k \to \Delta^n$ denote any face such that $x \circ d$ contains all vertices $x(j)$ and has no duplicates. Note that $x \circ d$ is an injection by (5). A degeneracy of $x \circ d$ has the same vertices as does x. Since X is ordered, x is a degeneracy of $x \circ d$. Hence, $x \circ d\colon \Delta^k \hookrightarrow X$ is the image of x.

Claim (7) follows from claims (2) and (6).

For claim (8) we first show that the maps $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple inclusions. To see this, suppose that $u, v \in X$ are vertices, T is a necklace, and $f\colon T \to B_{u,v}$ is a map; we want to show that f factors through X. Note that any simplex $\Delta^k \to B$ either factors through X or through Y. Suppose that the set of beads of T which factor through Y is non-empty (if it is empty, we are done). Take from it any maximal subset in which the beads are adjacent. We are left with a necklace $T' \subseteq T$ such that $f(T') \subseteq Y$ and $f(\alpha_{T'}), f(\omega_{T'}) \in X$. Since $\alpha_{T'}, \omega_{T'}$ are also in Y, they are in A, so the lifting property implies that $f(T') \subseteq A \subseteq X$. Hence f factors through X.

We have shown that $X \hookrightarrow B$ (and dually $Y \hookrightarrow B$) is a simple inclusion. Now we show that B is ordered, so suppose $u, v \in B$ are such that $u \preceq v$ and $v \preceq u$. There there are spines T and U and maps $T \to B_{u,v}$, $U \to B_{v,u}$. Consider the composite spine $T \vee U \to B_{u,u}$. If $u \in X$, then since $X \hookrightarrow B$ is a simple inclusion it follows that the image of $T \vee U$ maps entirely into X; so $u \preceq_X v$ and $v \preceq_X u$, which means $u = v$ because X is ordered. The same argument works if $u \in Y$, so this verifies antisymmetry of $\preceq_B$.

To verify condition (ii) of Definition 3.3, suppose $p, q\colon \Delta^k \to B$ are k-simplices with the same sequence of vertices; we wish to show $p = q$. We know that p factors through X or Y, and so does q; if both factor through Y, then the fact that Y is ordered implies that $p = q$ (similarly for X). So we may assume p factors through X and q factors through Y. By induction on k, the restrictions $p|_{\partial\Delta^k} = q|_{\partial\Delta^k}$ are equal, hence factor through A. By Lemma 3.5 applied to $A \hookrightarrow X$, the map p factors through A. Therefore it also factors through Y, and now we are done because q also factors through Y and Y is ordered. □

3.7. **Categorification of necklaces.** Let T be a necklace. Our next goal is to give a complete description of the simplicial category $\mathfrak{C}(T)$. The object set of this category is precisely T_0.

For vertices $a, b \in T_0$, let $V_T(a, b)$ denote the set of vertices in T between a and b, inclusive (with respect to the relation $\preceq$). Let $J_T(a, b)$ denote the union of $\{a, b\}$ with the set of joints between a and b. There is a unique subnecklace of T with joints $J_T(a, b)$ and vertices $V_T(a, b)$; let $\widetilde{B}_0, \widetilde{B}_1, \dots \widetilde{B}_\ell$ denote its beads. There are canonical inclusions of each $\widetilde{B}_i$ to T. Hence, there is a natural map

$$\mathfrak{C}(\widetilde{B}_k)(j_k, b) \times \mathfrak{C}(\widetilde{B}_{k-1})(j_{k-1}, j_k) \times \cdots \times \mathfrak{C}(\widetilde{B}_1)(j_1, j_2) \times \mathfrak{C}(\widetilde{B}_0)(a, j_1) \to \mathfrak{C}(T)(a, b)$$

obtained by first including the $\widetilde{B}_i$'s into T and then using the composition in $\mathfrak{C}(T)$. We will see that this map is an isomorphism. Note that each of the sets $\mathfrak{C}(\widetilde{B}_i)(-,-)$ has an easy description, as in Lemma 2.6); from this one may extrapolate a corresponding description for $\mathfrak{C}(T)(-,-)$, to be explained next.

Let $C_T(a, b)$ denote the poset whose elements are subsets of $V_T(a, b)$ which contain $J_T(a, b)$, ordered by inclusion. There is a pairing of categories

$$C_T(b, c) \times C_T(a, b) \to C_T(a, c)$$

given by union of subsets.

Applying the nerve functor, we obtain a simplicial category NC_T with object set T_0. For $a, b \in T_0$, an n-simplex in $NC_T(a, b)$ can be seen as a flag of sets $\overrightarrow{T} = T^0 \subseteq T^1 \subseteq \cdots \subseteq T^n$, where $J_T \subseteq T^0$ and $T^n \subseteq V_T$.

Proposition 3.8. *Let T be a necklace. There is a natural isomorphism of simplicial categories between $\mathfrak{C}(T)$ and NC_T.*

Proof. Write $T = B_1 \vee B_2 \vee \cdots \vee B_k$, where the B_i's are the beads of T. Then

$$\mathfrak{C}(T) = \mathfrak{C}(B_1) \amalg_{\mathfrak{C}(*)} \mathfrak{C}(B_2) \amalg_{\mathfrak{C}(*)} \cdots \amalg_{\mathfrak{C}(*)} \mathfrak{C}(B_k)$$

since $\mathfrak{C}$ preserves colimits. Note that $\mathfrak{C}(*) = \mathfrak{C}(\Delta^0) = *$, the category with one object and a single morphism (the identity).

Note that we have isomorphisms $\mathfrak{C}(B_i) \cong NC_{B_i}$ by Lemma 2.6. We therefore get maps of categories $\mathfrak{C}(B_i) \to NC_{B_i} \to NC_T$, and it is readily checked these extend to a map $f\colon \mathfrak{C}(T) \to NC_T$. To see that this functor is an isomorphism, it suffices to show that it is fully faithful (as it is clearly a bijection on objects).

For any $a, b \in T_0$ we will construct an inverse to the map $f\colon \mathfrak{C}(T)(a, b) \to NC_T(a, b)$, when $b > a$ (the case $b \leq a$ being obvious). Let B_r and B_s be the beads containing a and b, respectively (if a (resp. b) is a joint, let B_r (resp. B_s) be the latter (resp. former) of the two beads which contain it). Let $j_r, j_{r+1} \dots, j_{s+1}$ denote the elements of $J_T(a, b)$, indexed so that j_i and j_{i+1} lie in the bead B_i; note that $j_r = a$ and $j_{s+1} = b$.

Any simplex $x \in NC_T(a,b)_n$ can be uniquely written as the composite of n-simplices $x_s \circ \cdots \circ x_r$, where $x_i \in NC_T(j_i, j_{i+1})_n$. Now j_i and j_{i+1} are vertices within the same bead B_i of T, therefore x_i may be regarded as an n-simplex in $\mathfrak{C}(B_i)(j_i, j_{i+1})$. We then get associated n-simplices in $\mathfrak{C}(T)(j_i, j_{i+1})$, and taking their composite gives an n-simplex $\tilde{x} \in \mathfrak{C}(T)(a,b)$. We define a map $g\colon NC_T(a,b) \to \mathfrak{C}(T)(a,b)$ by sending x to $\tilde{x}$. One readily checks that this is well-defined and compatible with the simplicial operators, and it is also clear that $f \circ g = \mathrm{id}$.

To see that f is an isomorphism it suffices to now show that g is surjective. But upon pondering colimits of categories, it is clear that every map in $\mathfrak{C}(T)(a,b)$ can be written as a composite of maps from the $\mathfrak{C}(B_i)$'s. It follows at once that g is surjective. □

Corollary 3.9. *Let $T = B_0 \vee B_1 \vee \cdots \vee B_k$ be a necklace. Let $a, b \in T_0$ be such that $a < b$. Let $j_r, j_{r+1}, \ldots, j_{s+1}$ be the elements of $J_T(a,b)$ (in order), and let B_i denote the bead containing j_i and j_{i+1}, for $r \le i \le s$. Then the map*

$$\mathfrak{C}(B_s)(j_s, j_{s+1}) \times \cdots \times \mathfrak{C}(B_r)(j_r, j_{r+1}) \to \mathfrak{C}(T)(a,b)$$

is an isomorphism. Therefore $\mathfrak{C}(T)(a,b) \cong (\Delta^1)^N$ where $N = |V_T(a,b) - J_T(a,b)|$. In particular, $\mathfrak{C}(T)(a,b)$ is contractible if $a \le b$ and empty otherwise.

Proof. Follows at once from the previous lemma. □

Remark 3.10. Given a necklace T, there is a heuristic way to understand faces (both codimension one and higher) in the cubes $\mathfrak{C}(T)(a,b)$ in terms of "paths" from a to b in T. To choose a face in $\mathfrak{C}(T)(a,b)$, one chooses three subsets $Y, N, M \subset V_T(a,b)$ which cover the set $V_T(a,b)$ and are mutually disjoint. The set Y is the set of vertices which we require our path to go through – it must contain $J_T(a,b)$; the set N is the set of vertices which we require our path to not go through; and the set M is the set of vertices for which we leave the question open. Such choices determine a unique face in $\mathfrak{C}(T)(a,b)$. The dimension of this face is precisely the number of vertices in M.

4. The categorification functor

By this point, we fully understand $\mathfrak{C}(\Delta^n)$ as a simplicial category. Recall that $\mathfrak{C}\colon s\mathcal{S}et \to s\mathcal{C}at$ is defined for $S \in s\mathcal{S}et$ by the formula

$$\mathfrak{C}(S) = \operatorname*{colim}_{\Delta^n \to S} \mathfrak{C}(\Delta^n).$$

The trouble with this formula is that given a diagram $X\colon I \to s\mathcal{C}at$ of simplicial categories, it is generally quite difficult to understand the mapping spaces in the colimit. In our case, however, something special happens because the simplicial categories $\mathfrak{C}(\Delta^n)$ are "directed" in a certain sense. It turns out by making use of necklaces one can write down a precise description of the mapping spaces for $\mathfrak{C}(S)$; this is the goal of the present section.

Fix a simplicial set S and elements $a, b \in S_0$. For any necklace T and map $T \to S_{a,b}$, there is an induced map $\mathfrak{C}(T)(\alpha,\omega) \to \mathfrak{C}(S)(a,b)$. Let $(\mathcal{N}ec \downarrow S)_{a,b}$ denote the category whose objects are pairs $[T, T \to S_{a,b}]$ and whose morphisms are maps of necklaces $T \to T'$ giving commutative triangles over S. Then we obtain a map

$$\operatorname*{colim}_{T \to S \in (\mathcal{N}ec\downarrow S)_{a,b}} \Big[\mathfrak{C}(T)(\alpha,\omega)\Big] \longrightarrow \mathfrak{C}(S)(a,b). \tag{4.1}$$

Let us write $E_S(a,b)$ for the domain of this map. Note that there are composition maps

$$E_S(b,c) \times E_S(a,b) \longrightarrow E_S(a,c) \tag{4.2}$$

induced in the following way. Given $T \to S_{a,b}$ and $U \to S_{b,c}$ where T and U are necklaces, one obtains $T \vee U \to S_{a,c}$ in the evident manner. The composite

$$\begin{tikzcd}
\mathfrak{C}(U)(\alpha_U,\omega_U) \times \mathfrak{C}(T)(\alpha_T,\omega_T) \arrow[r] & \mathfrak{C}(T\vee U)(\omega_T,\omega_U) \times \mathfrak{C}(T\vee U)(\alpha_T,\omega_T) \arrow[d, "\mu"] \\
 & \mathfrak{C}(T\vee U)(\alpha_T,\omega_U)
\end{tikzcd}$$

induces the pairing of (4.2). One readily checks that E_S is a simplicial category with object set S_0, and (4.1) yields a map of simplicial categories $E_S \to \mathfrak{C}(S)$. Moreover, the construction E_S is clearly functorial in S.

Here is our first result:

Proposition 4.3. *For every simplicial set S, the map $E_S \to \mathfrak{C}(S)$ is an isomorphism of simplicial categories.*

Proof. First note that if S is itself a necklace then the identity map $S \to S$ is a terminal object in $(\mathcal{N}ec \downarrow S)_{a,b}$. It follows at once that $E_S(a,b) \to \mathfrak{C}(S)(a,b)$ is an isomorphism for all a and b.

Now let S be an arbitrary simplicial set, and choose vertices $a,b \in S_0$. We will show that $E_S(a,b) \to \mathfrak{C}(S)(a,b)$ is a bijection. Consider the commutative diagram of simplicial sets

$$\begin{tikzcd}
\Big(\operatorname{colim}_{\Delta^k\to S} E_{\Delta^k}\Big)(a,b) \arrow[r, "t"] \arrow[d, "\cong"'] & E_S(a,b) \arrow[d] \\
\Big(\operatorname{colim}_{\Delta^k\to S} \mathfrak{C}(\Delta^k)\Big)(a,b) \arrow[r, equal] & \mathfrak{C}(S)(a,b).
\end{tikzcd}$$

The bottom equality is the definition of $\mathfrak{C}$. The left-hand map is an isomorphism by our remarks in the first paragraph. It follows that the top map t is injective. To complete the proof it therefore suffices to show that t is surjective.

Choose an n-simplex $x \in E_S(a,b)_n$; it is represented by a necklace T, a map $f\colon T \to S_{a,b}$, and an element $\tilde{x} \in \mathfrak{C}(T)(\alpha,\omega)$. We have a commutative diagram

$$\begin{tikzcd}
\left(\operatorname{colim}_{\Delta^k\to T} \mathfrak{C}(\Delta^k)\right)(\alpha,\omega) \arrow[r] & \mathfrak{C}(T)(\alpha,\omega) \\
\left(\operatorname{colim}_{\Delta^k\to T} E_{\Delta^k}\right)(\alpha,\omega) \arrow[r] \arrow[u] \arrow[d, "f"'] & E_T(\alpha,\omega) \arrow[u] \arrow[d, "E_f"] \\
\left(\operatorname{colim}_{\Delta^k\to S} E_{\Delta^k}\right)(a,b) \arrow[r, "t"] & E_S(a,b).
\end{tikzcd}$$

The n-simplex in $E_T(\alpha,\omega)$ represented by $[T, \mathrm{id}_T\colon T \to T; \tilde{x}]$ is sent to x under E_f. It suffices to show that the middle horizontal map is surjective, for then x will be in the image of t. But the top map is an isomorphism, and the vertical arrows in the top row are isomorphisms by the remarks from the first paragraph. Thus, we are done. □

Corollary 4.4. *For any simplicial set S and elements $a, b \in S_0$, the simplicial set $\mathfrak{C}(S)(a,b)$ admits the following description. An n-simplex in $\mathfrak{C}(S)(a,b)$ consists of an equivalence class of triples $[T, T \to S, \overrightarrow{T}]$, where*

- *T is a necklace;*
- *$T \to S$ is a map of simplicial sets which sends α_T to a and ω_T to b; and*
- *$\overrightarrow{T}$ is a flag of sets $T^0 \subseteq T^1 \subseteq \cdots \subseteq T^n$ such that T^0 contains the joints of T and T^n is contained in the set of vertices of T.*

The equivalence relation is generated by considering $(T \to S; \overrightarrow{T})$ and $(U \to S; \overrightarrow{U})$ to be equivalent if there exists a map of necklaces $f\colon T \to U$ over S with $\overrightarrow{U} = f_*(\overrightarrow{T})$.

The ith face (resp. degeneracy) map omits (resp. repeats) the set T^i in the flag. That is, if $x = (T \to S; T^0 \subseteq \cdots \subseteq T^n)$ represents an n-simplex of $\mathfrak{C}(S)(a,b)$ and $0 \le i \le n$, then

$$s_i(x) = (T \to S; T^0 \subseteq \cdots \subseteq T^i \subseteq T^i \subseteq \cdots \subseteq T^n)$$

and

$$d_i(x) = (T \to S; T^0 \subseteq \cdots \subseteq T^{i-1} \subseteq T^{i+1} \subseteq \cdots \subseteq T^n).$$

Proof. This is a straightforward interpretation of the colimit appearing in the definition of E_S from (4.1). Recall that every colimit can be written as a coequalizer

$$\operatorname*{colim}_{T \to S \in (\mathcal{N}ec \downarrow S)_{a,b}} \Big[\mathfrak{C}(T)(\alpha,\omega)\Big] \cong \operatorname{coeq}\Big[\coprod_{T_1 \to T_2 \to S} \mathfrak{C}(T_1)(\alpha,\omega) \rightrightarrows \coprod_{T \to S} \mathfrak{C}(T)(\alpha,\omega)\Big],$$

and that elements of $\mathfrak{C}(T)$ are identified with flags of subsets of V_T, containing J_T, by Lemma 3.8. □

Our next goal is to simplify the equivalence relation appearing in Corollary 4.4 somewhat. To this end, let us introduce some terminology. A **flagged necklace** is a pair $[T, \overrightarrow{T}]$ where T is a necklace and $\overrightarrow{T}$ is a flag of subsets of V_T which all contain J_T. The **length of the flag** is the number of subset symbols, or one less than the number of subsets. A morphism of flagged necklaces $[T, \overrightarrow{T}] \to [U, \overrightarrow{U}]$ exists only if the flags have the same length, in which case it is a map of necklaces $f\colon T \to U$ such that $f(T^i) = U^i$ for all i. Finally, a flag $\overrightarrow{T} = (T^0 \subseteq \cdots \subseteq T^n)$ is called **flanked** if $T^0 = J_T$ and $T^n = V_T$. Note that if $[T, \overrightarrow{T}]$ and $[U, \overrightarrow{U}]$ are both flanked, then every morphism $[T, \overrightarrow{T}] \to [U, \overrightarrow{U}]$ is surjective (because its image will be a subnecklace of U having the same joints and vertices as U, hence it must be all of U).

Lemma 4.5. *Under the equivalence relation of Corollary 4.4, each of the triples $[T, T \to S, \overrightarrow{T}]$ is equivalent to one in which the flag is flanked. Moreover, two flanked triples are equivalent (in the sense of Corollary 4.4) if and only if they can be connected by a zig-zag of morphisms of flagged necklaces in which every triple of the zig-zag is flanked.*

Proof. Suppose given a flagged necklace $[T, T^0 \subseteq \cdots \subseteq T^n]$. There is a unique subnecklace $T' \hookrightarrow T$ whose set of joints is T^0 and whose vertex set is T^n. Then the pair $(T', T^0 \subseteq \cdots \subseteq T^n)$ is flanked. This assignment, which we call *flankification*, is actually functorial: a morphism of flagged necklaces $f\colon [T, \overrightarrow{T}] \to [U, \overrightarrow{U}]$ must map T' into U' and therefore gives a morphism $[T', \overrightarrow{T}] \to [U', \overrightarrow{U}]$.

Using the equivalence relation of Corollary 4.4, each triple $[T, T \to S, \overrightarrow{T}]$ will be equivalent to the flanked triple $[T', T' \to T \to S, \overrightarrow{T}]$ via the map $T' \to T$. If the flanked triple $[U, U \to S, \overrightarrow{U}]$ is equivalent to the flanked triple $[V, V \to S, \overrightarrow{V}]$ then there is a zig-zag of maps between triples which starts at the first and ends at the second, by Corollary 4.4. Applying the flankification functor gives a corresponding zig-zag in which every object is flanked. $\square$

Remark 4.6. By the previous lemma, we can alter our model for $\mathfrak{C}(S)(a, b)$ so that the n-simplices are equivalence classes of triples $[T, T \to S, \overrightarrow{T}]$ in which the flag is flanked, and the equivalence relation is given by maps (which are necessarily surjections) of flanked triples. Under this model the degeneracies and inner faces are given by the same description as before: repeating or omitting one of the subsets in the flag. The outer faces d_0 and d_n are now more complicated, however, because omitting the first or last subset in the flag may produce one which is no longer flanked; one must first remove the subset and then apply the flankification functor from Lemma 4.5. This model for $\mathfrak{C}(S)(a, b)$ was originally shown to us by Jacob Lurie; it will play only a very minor role in what follows.

Our next task will be to analyze surjections of flagged triples. Let T be a necklace and S a simplicial set. Say that a map $T \to S$ is **totally nondegenerate** if the image of each bead of T is a nondegenerate simplex of S. Note a totally nondegenerate map need not be an injection: for example, let $S = \Delta^1/\partial\Delta^1$ and consider the nondegenerate 1-simplex $\Delta^1 \to S$.

Recall that in a simplicial set S, if $z \in S$ is a degenerate simplex then there is a unique nondegenerate simplex z' and a unique degeneracy operator $s_\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ such that $z = s_\sigma(z')$; see [H, Lemma 15.8.4]. Using this, and the fact that degeneracy operators correspond to surjections of simplices, one finds that for any map $T \to S$ there is a necklace $\overline{T}$, a map $\overline{T} \to S$ which is totally nondegenerate, and a surjection of necklaces $T \to \overline{T}$ making the evident triangle commute; moreover, these three things are unique up to isomorphism.

Proposition 4.7. *Let S be a simplicial set and let $a, b \in S_0$.*

(a) Suppose that T and U are necklaces, $U \xrightarrow{u} S$ and $T \xrightarrow{t} S$ are two maps, and that t is totally nondegenerate. Then there is at most one surjection $U \xrightarrow{f} T$ such that $u = t \circ f$.

(b) Suppose that one has a diagram

$$\begin{tikzcd} U \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] & T \arrow[d] \\ V \arrow[r] & S \end{tikzcd}$$

where T, U, and V are flagged necklaces, $T \to S$ is totally nondegenerate, and f and g are surjections. Then there exists a unique map of flagged necklaces $V \to T$ making the diagram commute.

Proof. We first make the observation that if $A \to B$ is a surjection of necklaces and $B \neq *$ then every bead of B is surjected on by a unique bead of A. Also, each bead of A is either collapsed onto a joint of B or else mapped surjectively onto a bead of B.

For (a), note that we may assume $T \neq *$ (or else the claim is trivial). The necklace U can be written as the wedge of beads, some of which are sent to a point in S, call them bad, and some of which are not, call them good. We now use the above observation. For any surjection $f\colon U \to T$, each bad bead of U must be sent to a joint in T and each good bead of U must surject onto a unique bead of T, because $T \to S$ is totally non-degenerate.

This is almost enough to show that there is at most one surjection $U \to T$ over S; all that is left is to show that for each good bead B of U, there is at most one possible surjection to the corresponding bead C of T (over S). The image of B in S is a degeneracy of the image of C in S; since the latter is non-degenerate, the result follows by the uniqueness of degeneracies.

Next we turn to part (b). Note that the map $V \to T$ will necessarily be surjective, so the uniqueness part is guaranteed by (a); we need only show existence.

Observe that if B is a bead in U which maps to a point in V then it maps to a point in T, by the reasoning above. It now follows that there exists a necklace U', obtained by collapsing every bead of U that maps to a point in V, and a commutative diagram

$$\begin{tikzcd} U \arrow[rr, two heads, "f"] \arrow[dr, two heads] \arrow[dd, two heads, "g"'] & & T \arrow[dd] \\ & U' \arrow[ur, two heads, "f'"] \arrow[dl, two heads, "g'"'] & \\ V \arrow[rr] & & S \end{tikzcd}$$

Replacing U, f, and g by U', f', and g', and dropping the primes, we can now assume that g induces a one-to-one correspondence between beads of U and beads of V. Let $B_1, \ldots, B_m$ denote the beads of U, and let $C_1, \ldots, C_m$ denote the beads of V.

Since $f\colon U \to T$ is surjective, the image D_i of each bead B_i in U is sent either to a bead or a point in T. Hence we have $T = D_1 \vee D_2 \vee \cdots \vee D_m$. A surjective map of necklaces $V \to T$ is determined by a surjective map $C_i \to D_i$ for each i.

For each bead $C_i = \Delta^{c_i}$ in V, its image $z \in S$ is a degeneracy of a unique non-degenerate simplex $z' \in S$ by [H, Lemma 15.8.4]. In other words, there is a unique non-degenerate simplex $\Delta^{j_i} \to S$ and commutative diagram

$$\begin{tikzcd} \Delta^{c_i} \arrow[r, two heads, "!"] \arrow[d, "C_i"'] \arrow[dr, "z"] & \Delta^{j_i} \arrow[d, "!"] \\ V \arrow[r] & S \end{tikzcd}$$

It follows that $D_i = \Delta^{j_i}$, and we define the map $V \to T$ so that it sends C_i to D_i in the specified way.

Any map $V \to T$ induces a map from the vertices of V to the vertices of T, and one can check that our map will send joints of V to joints of T. Hence, it sends flags to flags, completing the proof.

$\square$

Corollary 4.8. *Let S be a simplicial set and $a, b \in S_0$. Under the equivalence relation from Corollary 4.4, every triple $[T, T \to S_{a,b}, \overrightarrow{T}]$ is equivalent to a unique triple $[U, U \to S_{a,b}, \overrightarrow{U}]$ which is both flanked and totally nondegenerate.*

Proof. Let $t = [T, T \to S_{a,b}, \overrightarrow{T}]$. Then t is clearly equivalent to at least one flanked, totally nondegenerate triple because we can replace t with $[T', T' \to S_{a,b}, \overrightarrow{T}]$ (flankification) and then with $[\overline{T'}, \overline{T'} \to S_{a,b}, \overrightarrow{T}']$ (defined above Proposition 4.7).

Now suppose that $[U, U \to S_{a,b}, \overrightarrow{U}]$ and $[V, V \to S_{a,b}, \overrightarrow{V}]$ are both flanked, totally nondegenerate, and equivalent in $\mathfrak{C}(S)(a,b)_n$. Then by Lemma 4.5 there is a zig-zag of maps between flanked necklaces (over S) connecting U to V:

$$\begin{tikzcd}
& W_1 \arrow[dl, two heads] \arrow[dr, two heads] & & W_2 \arrow[dl, two heads] \arrow[dr, two heads] & & \cdots \arrow[dl, two heads] \arrow[dr, two heads] & & W_k \arrow[dl, two heads] \arrow[dr, two heads] & \\
U = U_1 & & U_2 & & U_3 & \cdots & U_k & & U_{k+1} = V
\end{tikzcd}$$

Using Proposition 4.7, we inductively construct surjections of flanked necklaces $U_i \to U$ over S. This produces a surjection $V \to U$ over S. Similarly, we obtain a surjection $U \to V$ over S. By Proposition 4.7(a) these maps must be inverses of each other; that is, they are isomorphisms. $\square$

Remark 4.9. Again, as in Remark 4.6 the above corollary shows that we can describe $\mathfrak{C}(S)(a,b)$ as the simplicial set whose n-simplices are triples $[T, T \to S, \overrightarrow{T}]$ which are both flanked and totally nondegenerate. The degeneracies and inner faces are again easy to describe—they are repetition or omission of a set in the flag—but for the outer faces one must first omit a set and then apply flankification. This is just one description among many (see Remark **??**) and is more useful for thinking about examples than it is for proving results, because of these complications with the outer faces.

The following result is the culmination of our work in this section, and will turn out to be a key step in the proof of our main theorems. Fix a simplicial set S and vertices $a, b \in S_0$, and let F_n denote the category of flagged triples over $S_{a,b}$ that have length n. That is, the objects of F^n are triples $[T, T \to S_{a,b}, T^0 \subseteq \cdots \subseteq T^n]$ and morphisms are maps of necklaces $f \colon T \to T'$ over S such that $f(T^i) = (T')^i$ for all i.

Proposition 4.10. *For each $n \geq 0$, the nerve of F_n is homotopy discrete in $s\mathcal{S}et_K$.*

Proof. Recall from Lemma 4.5 that there is a functor $\phi \colon F_n \to F_n$ which sends any triple to its 'flankification'. There is a natural transformation from ϕ to the identity, and the image of ϕ is the subcategory $F_n' \subseteq F_n$ of flanked triples. It will therefore suffice to prove that (the nerve of) F_n' is homotopy discrete.

Recall from Corollary 4.8 that every component of F_n' contains a unique triple t which is both flanked and totally nondegenerate. Moreover, every triple in the same component as t admits a unique map to t—that is to say, t is a final object for its component. Therefore its component is contractible. This completes the proof. $\square$

4.11. **The functor $\mathfrak{C}$ applied to ordered simplicial sets.**

Note that even if a simplicial set S is small—say, in the sense that it has finitely many nondegenerate simplices—the space $\mathfrak{C}(S)(a,b)$ may be quite large. This is due to the fact that there are infinitely many necklaces mapping to S (if S is nonempty). For certain simplicial sets S, however, it is possible to restrict to necklaces which lie *inside of* S; this cuts down the possibilities. The following results and subsequent example demonstrate this. Recall the definition (3.3) of ordered simplicial sets.

Lemma 4.12. *Let D be an ordered simplicial set and let $a, b \in D_0$. Then every n-simplex in $\mathfrak{C}(D)(a,b)$ is represented by a unique triple $[T, T \to D, \overrightarrow{T}]$ in which T is a necklace, $\overrightarrow{T}$ is a flanked flag of length n, and the map $T \to D$ is injective.*

Proof. By Corollary 4.8, every n-simplex in $\mathfrak{C}(D)(a,b)$ is represented by a unique triple $[T, T \to D, \overrightarrow{T}]$ which is both flanked and totally non-degenerate. It suffices to show that if D is ordered, then any totally non-degenerate map $T \to D$ is injective. This follows from Lemma 3.6. □

Corollary 4.13. *Let D be an ordered simplicial set, and $a, b \in D_0$. Let $M_D(a,b)$ denote the simplicial set for which $M_D(a,b)_n$ is the set of triples $[T, T \xrightarrow{f} D_{a,b}, \overrightarrow{T}]$, where f is injective and $\overrightarrow{T}$ is a flanked flag of length n; face and boundary maps are as in Remark 4.6. Then there is a natural isomorphism*

$$\mathfrak{C}(D)(a,b) \xrightarrow{\cong} M_D(a,b).$$

Proof. This follows immediately from Lemma 4.12.

□

Example 4.14. Consider the simplicial set $S = \Delta^2 \amalg_{\Delta^1} \Delta^2$ depicted

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 1 \arrow[ur] \arrow[r] & 3 \end{tikzcd}$$

We will describe the mapping space $X = \mathfrak{C}(S)(0,3)$ by giving its non-degenerate simplices and face maps.

By Lemma 4.12, it suffices to consider flanked necklaces that inject into S. There are only five such necklaces that have endpoints 0 and 3. These are $T = \Delta^1 \vee \Delta^1$, which maps to S in two different ways f, g; and $U = \Delta^1 \vee \Delta^1 \vee \Delta^1$, $V = \Delta^1 \vee \Delta^2$, and $W = \Delta^2 \vee \Delta^1$, each of which maps uniquely into $S_{0,3}$. The image of T_0 under f is $\{0,1,3\}$ and under g is $\{0,2,3\}$. The images of U_0, V_0, and W_0 are all $\{0,1,2,3\}$.

We find that X_0 consists of three elements $[T; \{0,1,3\}]$, $[T; \{0,2,3\}]$ and $[U; \{0,1,2,3\}]$. There are two nondegenerate 1-simplices, $[V; \{0,1,3\} \subset \{0,1,2,3\}]$ and $[W; \{0,2,3\} \subset \{0,1,2,3\}]$. These connect the three 0-simplices in the obvious way, resulting in two one simplices with a common final vertex. There are no higher non-degenerate simplices. Thus $\mathfrak{C}(S)(0,3)$ looks like

$$\bullet \longleftarrow \bullet \longrightarrow \bullet.$$

5. Homotopical models for categorification

In the last section we gave a very explicit description of the mapping spaces $\mathfrak{C}(S)(a,b)$, for arbitrary simplicial sets S and $a, b \in S_0$. While this description was explicit, in some ways it is not very useful from a homotopical standpoint—in practice it is hard to use this description to identify the homotopy type of $\mathfrak{C}(S)(a,b)$.

In this section we will discuss a functor $\mathfrak{C}^{nec} \colon s\mathcal{S}et_J \to s\mathcal{C}at$ that has a simpler description than $\mathfrak{C}$ and which is more homotopical. We prove that for any simplicial set S there is a natural zigzag of weak equivalences between $\mathfrak{C}(S)$ and $\mathfrak{C}^{nec}(S)$. Variants of this construction are also introduced, leading to a collection of functors $s\mathcal{S}et \to s\mathcal{C}at$ all of which are weakly equivalent to $\mathfrak{C}$.

Let S be a simplicial set. A choice of $a, b \in S_0$ will be regarded as a map $\partial\Delta^1 \to S$. Let $(\mathcal{N}ec \downarrow S)_{a,b}$ be the overcategory for the inclusion functor $\mathcal{N}ec \hookrightarrow (\partial\Delta^1 \downarrow S)$. Finally, define

$$\mathfrak{C}^{nec}(S)(a,b) = N(\mathcal{N}ec \downarrow S)_{a,b}.$$

This is a simplicial category in an evident way.

Remark 5.1. Both the functor $\mathfrak{C}$ and the functor $\mathfrak{C}^{nec}$ have distinct advantages and disadvantages. The main advantage to $\mathfrak{C}$ is that it is left adjoint to the coherent nerve functor N (in fact it is a left Quillen functor); as such, it preserves colimits. However, as mentioned above, the functor $\mathfrak{C}$ can be difficult to use in practice because the mapping spaces have an awkward description.

It is at this point that our functor $\mathfrak{C}^{nec}$ becomes useful, because the mapping spaces are given as nerves of 1-categories. Many tools are available for determining when a morphism between nerves is a Kan equivalence. These tools are thus directly employable for determining weak equivalences in the Joyal model structure. We will give an example of this in Section 9 where we show that the unit map for the adjunction $(\mathfrak{C}, N)$ is a weak equivalence in the Joyal model structure.

Our main theorem is that there is a simple zigzag of weak equivalences between $\mathfrak{C}(S)$ and $\mathfrak{C}^{nec}(S)$; that is, there is a functor $\mathfrak{C}^{hoc}\colon s\mathcal{S}et \to s\mathcal{C}at$ and natural weak equivalences $\mathfrak{C} \leftarrow \mathfrak{C}^{hoc} \to \mathfrak{C}^{nec}$. We begin by describing the functor $\mathfrak{C}^{hoc}$.

Fix a simplicial set S. Define $\mathfrak{C}^{hoc}(S)$ to have object set S_0, and for every $a, b \in S_0$

$$\mathfrak{C}^{hoc}(S)(a,b) = \operatorname*{hocolim}_{T \in (\mathcal{N}ec \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha, \omega).$$

Note the similarities to Theorem 5.2, where it was shown that $\mathfrak{C}(S)(a,b)$ has a similar description in which the hocolim is replaced by the colim. In our definition of $\mathfrak{C}^{hoc}(S)(a,b)$ we mean to use a particular model for the homotopy colimit, namely the diagonal of the bisimplicial set whose (k,l)-simplices are pairs

$$(F\colon [k] \to (\mathcal{N}ec \downarrow S)_{a,b}; x \in \mathfrak{C}(F(0))(\alpha,\omega)_l),$$

where $F(0)$ denotes the necklace obtained by applying F to $0 \in [k]$ and then applying the forgetful functor $(\mathcal{N}ec \downarrow S)_{a,b} \to \mathcal{N}ec$. The composition law for $\mathfrak{C}^{hoc}$ is defined as in Lemma 3.8.

We proceed to establish natural transformations $\mathfrak{C}^{hoc} \to \mathfrak{C}^{nec}$ and $\mathfrak{C}^{hoc} \to \mathfrak{C}$. Note that $\mathfrak{C}^{nec}(S)(a,b)$ is the homotopy colimit of the constant functor $\{*\}\colon (\mathcal{N}ec \downarrow S)_{a,b} \to s\mathcal{S}et$ which sends everything to a point. The maps $\mathfrak{C}(T)(\alpha,\omega) \to *$ (where $T = F(0)$) induce our map $\mathfrak{C}^{hoc}(S)(a,b) \to \mathfrak{C}^{nec}(S)(a,b)$. Since the spaces $\mathfrak{C}(T)(\alpha,\omega)$ are all contractible simplicial sets (see Corollary 3.9), the induced map $\mathfrak{C}^{hoc}(S)(a,b) \to \mathfrak{C}^{nec}(S)(a,b)$ is a Kan equivalence. We thus obtain a natural weak equivalence of simplicial categories $\mathfrak{C}^{hoc}(S) \to \mathfrak{C}^{nec}(S)$.

For any diagram in a model category there is a canonical natural transformation from the homotopy colimit to the colimit of that diagram. Hence there is a morphism

$$\mathfrak{C}^{hoc}(S)(a,b) \to \operatorname*{colim}_{T \in (\mathcal{N}ec \downarrow S)_{a,b}} \mathfrak{C}(T)(\alpha,\omega) \cong \mathfrak{C}(S)(a,b).$$

(For the isomorphism we are using Proposition 4.3.) As this is natural in $a, b \in S_0$ and natural in S, we have a natural transformation $\mathfrak{C}^{hoc} \to \mathfrak{C}$.

Theorem 5.2. *For every simplicial set S, the maps $\mathfrak{C}(S) \leftarrow \mathfrak{C}^{hoc}(S) \to \mathfrak{C}^{nec}(S)$ defined above are weak equivalences of simplicial categories.*

Proof. We have already established that the natural transformation $\mathfrak{C}^{hoc} \to \mathfrak{C}^{nec}$ is an objectwise equivalence, so it suffices to show that for each simplicial set S and objects $a, b \in S_0$ the natural map $\mathfrak{C}^{hoc}(S)(a, b) \to \mathfrak{C}(S)(a, b)$ is also a Kan equivalence.

Recall that $\mathfrak{C}^{hoc}(S)(a, b)$ is the diagonal of a bisimplicial set whose lth 'horizontal' row is the nerve NF_l of the category of flagged necklaces mapping to S, where the flags have length l. Also recall from Corollary 4.4 that $\mathfrak{C}(S)(a, b)$ is the simplicial set which in level l is $\pi_0(NF_l)$. But Proposition 4.10 says that $NF_l \to \pi_0(NF_l)$ is a Kan equivalence, for every l. It follows that $\mathfrak{C}^{hoc}(S)(a, b) \to \mathfrak{C}(S)(a, b)$ is also a Kan equivalence. $\square$

5.3. **Other models for categorification.** One can imagine variations of our basic construction in which one replaces necklaces with other convenient simplicial sets—which we might term "gadgets", for lack of a better word. We will see in Section 6, for instance, that using *products* of necklaces leads to a nice theorem about the categorification of a product. Later, in Section 9, several key arguments will hinge on a clever choice of what gadgets to use. In the material below we give some basic requirements of the "gadgets" which will ensure they give a model equivalent to that of necklaces.

Suppose $\mathcal{P}$ is a subcategory of $s\mathcal{S}et_{*,*} = (\partial\Delta^1 \downarrow s\mathcal{S}et)$ containing the terminal object. For any simplicial set S and vertices $a, b \in S_0$, let $(\mathcal{P} \downarrow S)_{a,b}$ denote the overcategory whose objects are pairs $[P, P \to S]$, where $P \in \mathcal{P}$ and the map $P \to S$ sends $\alpha \mapsto a$ and $\omega \mapsto b$. Define

$$\mathfrak{C}^{\mathcal{P}}(S)(a, b) = N(\mathcal{P} \downarrow S)_{a,b}.$$

The object $\mathfrak{C}^{\mathcal{P}}$ is simply an assignment which takes a simplicial set S with two distinguished vertices and produces a "$\mathcal{P}$-mapping space." However, if $\mathcal{P}$ is closed under the wedge operation (i.e. for any $P_1, P_2 \in \mathcal{P}$ one has $P_1 \vee P_2 \in \mathcal{P}$), then $\mathfrak{C}^{\mathcal{P}}$ may be given the structure of a functor $s\mathcal{S}et \to s\mathcal{C}at$ in the evident way.

Definition 5.4. *We call a subcategory $\mathcal{G} \subseteq s\mathcal{S}et_{*,*}$ a* ***category of gadgets*** *if it satisfies the following properties:*

(1) $\mathcal{G}$ contains the category $\mathcal{N}ec$,
(2) For every object $X \in \mathcal{G}$ and every necklace T, all maps $T \to X$ are contained in $\mathcal{G}$, and
(3) For any $X \in \mathcal{G}$, the simplicial set $\mathfrak{C}(X)(\alpha, \omega)$ is contractible.

The category $\mathcal{G}$ is said to be ***closed under wedges*** *if it is also true that*

(4) For any $X, Y \in \mathcal{G}$, the wedge $X \vee Y$ also belongs to $\mathcal{G}$.

The above definition can be generalized somewhat by allowing $\mathcal{N}ec \to \mathcal{G}$ to be an arbitrary functor over a natural transformation in $s\mathcal{S}et$; we do not need this generality in the present paper.

Proposition 5.5. *Let $\mathcal{G}$ be a category of gadgets. Then for any simplicial set S and any $a, b \in S_0$, the natural map*

$$\mathfrak{C}^{nec}(S)(a, b) \longrightarrow \mathfrak{C}^{\mathcal{G}}(S)(a, b)$$

(induced by the inclusion $\mathcal{N}ec \hookrightarrow \mathcal{G}$) is a Kan equivalence. If $\mathcal{G}$ is closed under wedges then the map of simplicial categories $\mathfrak{C}^{nec}(S) \to \mathfrak{C}^{\mathcal{G}}(S)$ is a weak equivalence.

Proof. Let $j\colon (\mathcal{N}ec \downarrow S)_{a,b} \to (\mathcal{G} \downarrow S)_{a,b}$ be the functor induced by the inclusion map $\mathcal{N}ec \hookrightarrow \mathcal{G}$. The map in the statement of the proposition is just the nerve of j. To verify that it is a Kan equivalence, it is enough by Quillen's Theorem A [Q] to verify that all the overcategories of j are contractible. So fix an object $[X, X \to S]$ in $(\mathcal{G} \downarrow S)_{a,b}$. The overcategory $(j \downarrow [X, X \to S])$ is precisely the category $(\mathcal{N}ec \downarrow X)_{\alpha,\omega}$, the nerve of which is $\mathfrak{C}^{nec}(X)(\alpha, \omega)$. By Theorem 5.2 and our assumptions on $\mathcal{G}$, this is contractible.

The second statement of the result is a direct consequence of the first. □

6. Properties of categorification

In this section we establish two main properties of the categorification functor $\mathfrak{C}$. First, we prove that there is a natural weak equivalence $\mathfrak{C}(X \times Y) \simeq \mathfrak{C}(X) \times \mathfrak{C}(Y)$. Second, we prove that whenever $S \to S'$ is a weak equivalence in $s\mathcal{S}et_J$ it follows that $\mathfrak{C}(S) \to \mathfrak{C}(S')$ is a weak equivalence in $s\mathcal{C}at$. These properties are also proven in [L], but the proofs we give here are of a different nature and make central use of the $\mathfrak{C}^{nec}$ functor.

If $T_1, \ldots, T_n$ are necklaces then they are, in particular, ordered simplicial sets in the sense of Definition 3.3. So $T_1 \times \cdots \times T_n$ is also ordered, by Lemma 3.6. Let $\mathcal{G}$ be the full subcategory of $s\mathcal{S}et_{*,*} = (\partial\Delta^1 \downarrow s\mathcal{S}et)$ whose objects are products of necklaces with a map $f\colon \partial\Delta^1 \to T_1 \times \cdots \times T_n$ that has $f(0) \preceq f(1)$.

Proposition 6.1. *The category $\mathcal{G}$ is a category of gadgets in the sense of Definition 5.4.*

For the proof of this one needs to verify that $\mathfrak{C}(T_1 \times \cdots \times T_n)(\alpha, \omega) \simeq *$. This is not difficult, but is a bit of a distraction; we prove it later as Proposition A.4.

Proposition 6.2. *For any simplicial sets X and Y, both $\mathfrak{C}(X \times Y)$ and $\mathfrak{C}(X) \times \mathfrak{C}(Y)$ are simplicial categories with object set $X_0 \times Y_0$. For any $a_0, b_0 \in X$ and $a_1, b_1 \in Y$, the natural map*

$$\mathfrak{C}(X \times Y)(a_0a_1, b_0b_1) \to \mathfrak{C}(X)(a_0, b_0) \times \mathfrak{C}(Y)(a_1, b_1)$$

induced by $\mathfrak{C}(X \times Y) \to \mathfrak{C}(X)$ and $\mathfrak{C}(X \times Y) \to \mathfrak{C}(Y)$ is a Kan equivalence. Consequently, the map of simplicial categories

$$\mathfrak{C}(X \times Y) \to \mathfrak{C}(X) \times \mathfrak{C}(Y)$$

is a weak equivalence in $s\mathcal{C}at$.

Proof. Let $\mathcal{G}$ denote the above category of gadgets, in which the objects are products of necklaces. By Theorem 5.2 and Proposition 5.5 it suffices to prove the result for $\mathfrak{C}^{\mathcal{G}}$ in place of $\mathfrak{C}$.

Consider the functors

$$(\mathcal{G} \downarrow X \times Y)_{a_0a_1, b_0b_1} \underset{\theta}{\overset{\phi}{\rightleftarrows}} (\mathcal{G} \downarrow X)_{a_0,b_0} \times (\mathcal{G} \downarrow Y)_{a_1,b_1}$$

given by

$$\phi\colon [G, G \to X \times Y] \mapsto \big([G, G \to X \times Y \to X], [G, G \to X \times Y \to Y]\big)$$

and
$$\theta\colon \big([G, G \to X], [H, H \to Y]\big) \mapsto [G \times H, G \times H \to X \times Y].$$
Note that we are using that the subcategory $\mathcal{G}$ is closed under finite products.

It is very easy to see that there is a natural transformation $\mathrm{id} \to \theta\phi$, obtained by using diagonal maps, and a natural transformation $\phi\theta \to \mathrm{id}$, obtained by using projections. As a consequence, the maps θ and ϕ induce inverse homotopy equivalences on the nerves. This completes the proof.

□

Let $E\colon \mathcal{S}et \to s\mathcal{S}et$ denote the 0-coskeleton functor (see [AM]). For any simplicial set X and set S, we have $\mathrm{Hom}(X, ES) = \mathrm{Hom}(X_0, S)$. In particular, if $n \in \mathbb{N}$ we denote $E^n = E\{0, 1, \ldots, n\}$.

Lemma 6.3. *For any $n \geq 0$, the simplicial category $\mathfrak{C}(E^n)$ is contractible in $s\mathcal{C}at$—that is to say, all the mapping spaces in $\mathfrak{C}(E^n)$ are contractible.*

Proof. By Theorem 5.2 it is sufficient to prove that the mapping space $\mathfrak{C}^{nec}(E^n)(i, j)$ is contractible, for every $i, j \in \{0, 1, \ldots, n\}$. This mapping space is the nerve of the overcategory $(\mathcal{N}ec \downarrow E^n)_{i,j}$.

Observe that if T is a necklace then any map $T \to E^n$ extends uniquely over $\Delta[T]$. This is because maps into E^n are determined by what they do on the 0-skeleton, and $T \hookrightarrow \Delta[T]$ is an isomorphism on 0-skeleta.

Consider two functors
$$f, g\colon (\mathcal{N}ec \downarrow E^n)_{i,j} \to (\mathcal{N}ec \downarrow E^n)_{i,j}$$
given by
$$f\colon [T, T \xrightarrow{x} E^n] \mapsto [\Delta[T], \Delta[T] \xrightarrow{\bar{x}} E^n] \text{ and } g\colon [T, T \xrightarrow{x} E^n] \mapsto [\Delta^1, \Delta^1 \xrightarrow{z} E^n].$$
Here $\bar{x}$ is the unique extension of x to $\Delta[T]$, and z is the unique 1-simplex of E^n connecting i to j. Observe that g is a constant functor.

It is easy to see that there are natural transformations $\mathrm{id} \to f \leftarrow g$. The functor g factors through the terminal category $\{*\}$, so after taking nerves the identity map is null homotopic. Hence $(\mathcal{N}ec \downarrow E^n)_{i,j}$ is contractible.

□

For completeness (and because it is short) we include the following lemma, established in [L, Proof of 2.2.5.1]:

Lemma 6.4. *The functor $\mathfrak{C}\colon s\mathcal{S}et \to s\mathcal{C}at$ preserves cofibrations.*

Sketch of proof. Every cofibration in $s\mathcal{S}et$ is obtained by compositions and cobase changes from boundary inclusions of simplices. It therefore suffices to show that for each $n \geq 0$ the map $f\colon \mathfrak{C}(\partial\Delta^n) \to \mathfrak{C}(\Delta^n)$ is a cofibration in $s\mathcal{C}at$. Let $0 \leq i, j \leq n$. If $i > 0$ or $j < n$ then $(\mathcal{N}ec \downarrow \Delta^n)_{i,j} \cong (\mathcal{N}ec \downarrow \partial\Delta^n)_{i,j}$, whereby
$$f(i, j)\colon \mathfrak{C}(\partial\Delta^n)(i, j) \to \mathfrak{C}(\Delta^n)(i, j)$$
is an isomorphism by Proposition 4.3. For the remaining case $i = 0, j = n$, the map $f(0, n)$ is the inclusion of the boundary of a cube $b\colon \partial((\Delta^1)^{n-1}) \to (\Delta^1)^{n-1}$.

Let $U\colon s\mathcal{S}et \to s\mathcal{C}at$ denote the functor which sends a simplicial set S to the unique simplicial category $U(S)$ with two objects x, y and morphisms $\mathrm{Hom}(x, x) = \mathrm{Hom}(y, y) = \{*\}$, $\mathrm{Hom}(y, x) = \emptyset$, and $\mathrm{Hom}(x, y) = S$. In view of the generating cofibrations for $s\mathcal{C}at$ (see [B]), it is easy to show that U preserves cofibrations.

Hence $U(b)$ is a cofibration. Notice that f is the pushout of $U(b)$ along the obvious map $U[\partial((\Delta^1)^{n-1})] \to \mathfrak{C}(\partial\Delta^n)$ sending $x \mapsto 0$ and $y \mapsto n$. Thus, f is a cofibration. $\square$

Lemma 6.5. *The coherent nerve functor $N\colon s\mathcal{C}at \to s\mathcal{S}et_J$ preserves fibrant objects.*

Sketch of proof. Fibrant objects in $s\mathcal{S}et_J$ are characterized as those objects that have the right lifting property with respect to inner horn inclusions $f^{n,k}\colon \Lambda^n_k \to \Delta^n$, where $0 < k < n$. Hence, it suffices to show that each $\mathfrak{C}(f^{n,k})$ is an acyclic cofibration. This calculation is similar to the one in the proof of Lemma 6.4. $\square$

In the following proposition, we use the terminology from the Joyal model structure (see Remark 2.4).

Proposition 6.6. *If $S \to S'$ is a weak equivalence in $s\mathcal{S}et_J$ then $\mathfrak{C}(S) \to \mathfrak{C}(S')$ is a weak equivalence of simplicial categories.*

Proof. For any simplicial set X, the map $\mathfrak{C}(X \times E^n) \to \mathfrak{C}(X)$ induced by projection is a weak equivalence in $s\mathcal{C}at$. This follows by combining Proposition 6.2 with Lemma 6.3:

$$\mathfrak{C}(X \times E^n) \xrightarrow{\sim} \mathfrak{C}(X) \times \mathfrak{C}(E^n) \xrightarrow{\sim} \mathfrak{C}(X).$$

Since $X \amalg X \hookrightarrow X \times E^1$ is a cofibration in $s\mathcal{S}et$, $\mathfrak{C}(X) \amalg \mathfrak{C}(X) = \mathfrak{C}(X \amalg X) \to \mathfrak{C}(X \times E^1)$ is a cofibration in $s\mathcal{C}at$, by Lemma 6.4. It follows that $\mathfrak{C}(X \times E^1)$ is a cylinder object for $\mathfrak{C}(X)$ in $s\mathcal{C}at$. So if $\mathcal{D}$ is a fibrant simplicial category we may compute homotopy classes of maps $[\mathfrak{C}(X), \mathcal{D}]$ as the coequalizer

$$\operatorname{coeq}\Big(s\mathcal{C}at(\mathfrak{C}(X \times E^1), \mathcal{D}) \rightrightarrows s\mathcal{C}at(\mathfrak{C}(X), \mathcal{D})\Big).$$

But using the adjunction, this is isomorphic to

$$\operatorname{coeq}\Big(s\mathcal{S}et(X \times E^1, N\mathcal{D}) \rightrightarrows s\mathcal{S}et(X, N\mathcal{D})\Big).$$

Since $\mathcal{D}$ is fibrant in $s\mathcal{C}at$, $N\mathcal{D}$ is fibrant in $s\mathcal{S}et_J$ by Lemma 6.5. So the above coequalizer is $[X, N\mathcal{D}]$, and we have identified

$$[\mathfrak{C}(X), \mathcal{D}] \cong [X, N\mathcal{D}]. \tag{6.7}$$

Now let $S \to S'$ be a weak equivalence in $s\mathcal{S}et_J$. Then $\mathfrak{C}(S) \to \mathfrak{C}(S')$ is a map between cofibrant objects of $s\mathcal{C}at$. To prove that it is a weak equivalence in $s\mathcal{C}at$ it is sufficient to prove that the induced map on homotopy classes

$$[\mathfrak{C}(S'), \mathcal{D}] \to [\mathfrak{C}(S), \mathcal{D}]$$

is a bijection, for every fibrant object $\mathcal{D} \in s\mathcal{C}at$. Since $N\mathcal{D}$ is fibrant and $S \to S'$ is a weak equivalence, we have that $[S', N\mathcal{D}] \to [S, N\mathcal{D}]$ is a bijection; the result then follows by (6.7). $\square$

Corollary 6.8. *The functors $\mathfrak{C}\colon s\mathcal{S}et_J \rightleftarrows s\mathcal{C}at\colon N$ are a Quillen pair.*

Proof. The functor $\mathfrak{C}$ preserves cofibrations by Lemma 6.4 and preserves weak equivalences by Proposition 6.6. $\square$

7. Background on mapping spaces in model categories

Given two objects X and Y in a model category $\mathcal{M}$, there is an associated simplicial set $\mathrm{hMap}_{\mathcal{M}}(X,Y)$ called a "homotopy function complex" from X to Y. The basic theory of these function complexes is due to Dwyer-Kan [DK1, DK2, DK3]. As recounted in the introduction, there are several different ways to write down models for these function complexes, all of which turn out to be weakly equivalent. In this section we give a brief review of some of this machinery.

7.1. **Mapping spaces via cosimplicial resolutions.** Let $\mathcal{M}$ be a model category, and let $c\mathcal{M}$ be the Reedy model category of cosimplicial objects in $\mathcal{M}$ [H, Chapter 15]. For any $X \in \mathcal{M}$ we will write cX for the constant cosimplicial object consising of X in every dimension, where every coface and codegeneracy is the identity.

If $X \in \mathcal{M}$, a **cosimplicial resolution** of X is a Reedy cofibrant replacement $Q^\bullet \xrightarrow{\sim} cX$. Given such a cosimplicial resolution and an object $Z \in \mathcal{M}$, we may form the simplicial set $\mathcal{M}(Q^\bullet, Z)$ given by

$$[n] \mapsto \mathcal{M}(Q^n, Z).$$

It is known [H, 16.5.5] that if $Z \to Z'$ is a weak equivalence between fibrant objects then the induced map $\mathcal{M}(Q^\bullet, Z) \to \mathcal{M}(Q^\bullet, Z')$ is a Kan equivalence of simplicial sets.

7.2. **Mapping spaces via nerves of categories.** For any object $X \in \mathcal{M}$, let $\mathcal{Q}(X)$ be the category whose objects are pairs $[Q, Q \to X]$ where Q is cofibrant and $Q \to X$ is a weak equivalence. For any object $Y \in \mathcal{M}$, there is a functor

$$\mathcal{M}(-,Y)\colon \mathcal{Q}(X)^{op} \longrightarrow \mathcal{S}et$$

sending $[Q, Q \to X]$ to $\mathcal{M}(Q,Y)$. We can regard this functor as taking values in $s\mathcal{S}et$ by composing with the embedding $\mathcal{S}et \hookrightarrow s\mathcal{S}et$.

Consider the simplicial set $\mathrm{hocolim}_{\mathcal{Q}(X)^{op}} \mathcal{M}(-,Y)$. We fix our model for the hocolim functor to be the result of first taking the simplicial replacement of a diagram and then applying geometric realization. Notice in our case that in dimension n the simplicial replacement consists of diagrams of weak equivalences $Q_0 \leftarrow Q_1 \leftarrow \cdots \leftarrow Q_n$ over X (with each $Q_i \to X$ in $\mathcal{Q}(X)$), together with a map $Q_0 \to Y$. This shows that the simplicial replacement is nothing but the nerve of the category whose objects are zig-zags $[X \xleftarrow{\sim} Q \to Y]$ where Q is cofibrant and $Q \to X$ is a weak equivalence; a map from $[X \xleftarrow{\sim} Q \to Y]$ to $[X \xleftarrow{\sim} Q' \to Y]$ is a map $Q \to Q'$ making the evident diagram commute.

Categories of zig-zags like the one considered above were first studied in [DK3]. There are many variations, and it is basically the case that all sensible variations have weakly equivalent nerves; moreover, these nerves are weakly equivalent to the homotopy function complex $\mathrm{hMap}(X,Y)$ (defined to be the space of maps in the simplicial localization of $\mathcal{M}$ with respect to the weak equivalences). We will next recall some of this machinery. In addition to [DK3], see [D1].

Following [DK3], write $(\mathrm{Wcofib})^{-1}\mathcal{M}(\mathrm{Wfib})^{-1}(X,Y)$ to denote the category whose objects are zig-zags

$$X \overset{\sim}{\twoheadleftarrow} U \longrightarrow V \overset{\sim}{\leftarrowtail} Y,$$

and where the maps are natural transformations of diagrams which are the identity on X and on Y. Similarly, let $W^{-1}\mathcal{M}W^{-1}(X,Y)$ be the category whose objects are zig-zags

$$X \xleftarrow{\sim} U \longrightarrow V \xleftarrow{\sim} Y,$$

let $\mathcal{M}(\mathrm{Wfib})^{-1}\mathcal{M}(X,Y)$ be the category whose objects are zig-zags

$$X \longrightarrow U \overset{\sim}{\twoheadleftarrow} V \longrightarrow Y,$$

and so on.

Note that there are natural inclusions of two types: an example of the first is $\mathcal{M}(\mathrm{Wfib})^{-1}(X,Y) \hookrightarrow \mathcal{M}\,\mathrm{W}^{-1}(X,Y)$ (induced by $\mathrm{Wfib} \hookrightarrow \mathrm{W}$), and an example of the second is

$$\mathcal{M}\,\mathrm{W}^{-1}(X,Y) \hookrightarrow \mathcal{M}\,\mathrm{W}^{-1}\,\mathcal{M}(X,Y)$$

which sends

$$[X \xleftarrow{\sim} A \longrightarrow Y] \mapsto [X \xrightarrow{\mathrm{id}} X \xleftarrow{\sim} A \longrightarrow Y].$$

The following proposition is a very basic one in this theory, and will be used often in the remainder of the paper; we have included the proof for completeness, and because it is simple.

Proposition 7.3. *When Y is fibrant, the maps in the following commutative square all induce weak equivalences on nerves:*

$$\begin{tikzcd} \mathcal{M}(\mathrm{Wfib})^{-1}(X,Y) \arrow[r, tail] \arrow[d, tail] & \mathcal{M}(\mathrm{Wfib})^{-1}\mathcal{M}(X,Y). \arrow[d, tail] \\ \mathcal{M}W^{-1}(X,Y) \arrow[r, tail] & \mathcal{M}W^{-1}\mathcal{M}(X,Y) \end{tikzcd}$$

Proof. Denote all the inclusions in the square by j.

We start with the left vertical map. Given a zig-zag $[X \xleftarrow{\sim} A \longrightarrow Y]$, functorially factor the map $A \to X \times Y$ as $A \overset{\sim}{\rightarrowtail} P \twoheadrightarrow X \times Y$. Since Y is fibrant the projection $X \times Y \to X$ is a fibration, and so the composite $P \to X$ is a fibration as well. Define a functor $F\colon \mathcal{M}\,\mathrm{W}^{-1}(X,Y) \to \mathcal{M}(\mathrm{Wfib})^{-1}(X,Y)$ by sending $[X \xleftarrow{\sim} A \longrightarrow Y]$ to $[X \overset{\sim}{\twoheadleftarrow} P \longrightarrow Y]$. There are natural transformations $\mathrm{id} \to j \circ F$ and $\mathrm{id} \to F \circ j$, which shows that on nerves F and j are homotopy inverses.

A very similar proof works for the right vertical map in the diagram. Given a zig-zag $[X \longrightarrow U \xleftarrow{\sim} V \longrightarrow Y]$, functorially factor $V \to U \times Y$ as $V \overset{\sim}{\rightarrowtail} P \twoheadrightarrow U \times Y$. Define $F\colon \mathcal{M}W^{-1}\mathcal{M}(X,Y) \to \mathcal{M}(\mathrm{Wfib})^{-1}\mathcal{M}(X,Y)$ by sending

$$[X \longrightarrow U \xleftarrow{\sim} V \longrightarrow Y] \mapsto [X \longrightarrow U \overset{\sim}{\twoheadleftarrow} P \longrightarrow Y].$$

This gives a homotopy inverse for j.

For the top horizontal map we do not even need to use that Y is fibrant. Define a homotopy inverse by sending $[X \longrightarrow U \overset{\sim}{\twoheadleftarrow} V \longrightarrow Y]$ to the zig-zag $[X \overset{\sim}{\twoheadleftarrow} P \longrightarrow Y]$ where P is the pullback of $X \longrightarrow U \overset{\sim}{\twoheadleftarrow} V$.

Finally, the bottom horizontal map induces a weak equivalence on nerves because the other three maps do. $\square$

Let $QX^\bullet \to X$ be a cosimplicial resolution of X in $c\mathcal{M}$. Following [DK3], we now relate the simplicial set $\mathcal{M}(QX^\bullet, Y)$ to the nerves of the categories of zig-zags considered above.

For any simplicial set K, let $\mathbf{\Delta} K$ be the category of simplices of K. This is none other than the overcategory $(S \downarrow K)$, where $S\colon \mathbf{\Delta} \to s\mathcal{S}et$ is the functor $[n] \mapsto \Delta^n$.

It is known that the nerve of $\mathbf{\Delta}K$ is naturally weakly equivalent to K (see [D1, text prior to Prop. 2.4] for an explanation).

There is a functor $\mathbf{\Delta}\mathcal{M}(QX^\bullet, Y) \to \mathcal{M}(\mathrm{Wfib})^{-1}(X,Y)$ sending $([n], QX^n \to Y)$ to $[X \overset{\sim}{\twoheadleftarrow} QX^n \longrightarrow Y]$.

Proposition 7.4. *Let $QX^\bullet \to X$ be a Reedy cofibrant resolution of X. Then $\mathbf{\Delta}\mathcal{M}(QX^\bullet, Y) \to \mathcal{M}(\mathrm{Wfib})^{-1}(X,Y)$ induces a weak equivalence on nerves.*

Proof. The result is proven in [DK3], but see also [D1, Thm. 2.4]. □

Remark 7.5. To briefly summarize the main points of this section, we have that when Y is fibrant the following maps of categories all induce weak equivalences on the nerves:

$$\mathbf{\Delta}\mathcal{M}(QX^\bullet, Y) \longrightarrow \mathcal{M}\,\mathrm{Wfib}^{-1}(X,Y) \rightarrowtail \mathcal{M}\,\mathrm{W}^{-1}(X,Y) \rightarrowtail \mathcal{M}\,\mathrm{W}^{-1}\,\mathcal{M}(X,Y)$$

In particular, the nerves all have the homotopy type of the homotopy function complex $\mathrm{hMap}(X,Y)$.

8. Dwyer-Kan models for quasi-category mapping spaces

In this section we give some other models for the mapping spaces in a quasicategory. These have the advantage of being relatively easy to compute. However, they have the disadvantage that they do not admit a composition law.

8.1. **The canonical cosimplicial framing on $s\mathcal{S}et_J$.** Let $E\colon \mathcal{S}et \to s\mathcal{S}et$ denote the 0-coskeleton functor, as in Section 6. For a set S, we may also describe ES as the nerve of the groupoid $E_G S$ with object set S and a single morphism $a \to b$ for each $a, b \in S$.

Lemma 8.2. *For any nonempty set S, the map $ES \to \Delta^0$ is an acyclic fibration in $s\mathcal{S}et_J$.*

Proof. The acyclic fibrations in the Joyal model category $s\mathcal{S}et_J$ are the same as those in the Kan model category $s\mathcal{S}et_K$, as both model categories have the same cofibrations. It is easy to check that $ES \to \Delta^0$ has the right lifting property with respect to the maps $\partial\Delta^n \to \Delta^n$. □

The forgetful functor $\mathbf{\Delta} \hookrightarrow \mathcal{S}et$ describes a cosimplicial set whose nth object is $[n] = \{0, 1, \ldots, n\}$. Applying the functor E gives a cosimplicial object $[n] \mapsto E^n = N(E_G([n]))$ in $s\mathcal{S}et_J$. It is easy to check that $E^\bullet$ is Reedy cofibrant in $c(s\mathcal{S}et_J)$, and the above lemma shows that each E^n is contractible in $s\mathcal{S}et_J$. Note that the evident inclusion of categories $[n] \hookrightarrow E_G([n])$ induces a levelwise cofibration $\Delta^\bullet \hookrightarrow E^\bullet$.

8.3. **Three cosimplicial versions of Δ^1.**

In his book, Lurie at various times uses three internal models for the mapping space between vertices in a simplicial set S. They are called Hom_S^R, Hom_S^L, and Hom_S; the descriptions of all of these can be found in [L, Section 1.2.2]. In this subsection, we show that each of these can be understood as the mapping spaces coming from various cosimplicial resolutions of Δ^1. We also give one new model, Hom_S^E. See Remark 8.8.

Recall that for any simplicial sets M and N, the join $M \star N$ is a simplicial set with

$$(M \star N)_n = \coprod_{-1 \le i \le n} M_i \times N_{n-i-1},$$

where we put $M_{-1} = N_{-1} = \Delta^0$ (see [L, 1.2.8.1]). Note that $\star$ is a bifunctor and that $M \star \emptyset = \emptyset \star M = M$, so in particular there are natural inclusions $M \hookrightarrow M \star N$ and $N \hookrightarrow M \star N$.

We let $C_R(M)$ and $C_L(M)$ denote the quotient $(M \star \Delta^0)/M$ and $(\Delta^0 \star M)/M$, respectively. For any simplicial set M, let $C_{cyl}(M)$ be the pushout

$$\partial\Delta^1 \leftarrow M \times \partial\Delta^1 \hookrightarrow M \times \Delta^1.$$

Note that there is a natural inclusion $C_R(M) \hookrightarrow C_{cyl}(M)$, and a natural surjection $C_{cyl}(M) \to C_R(M)$ exhibiting $C_R(M)$ as a retract of $C_{cyl}(M)$. The same can be said if we replace C_R with C_L.

Let $C_R^\bullet$ (respectively $C_L^\bullet$) denote the cosimplicial space $[n] \mapsto C_R(\Delta^n)$ (resp. $[n] \mapsto C_L(\Delta^n)$). Write $C_{cyl}^\bullet$ for the cosimplicial space $[n] \mapsto C_{cyl}(\Delta^n)$. Finally, write $C_E^\bullet$ for the cosimplicial space obtained as the pushout of

$$\partial\Delta^1 \leftarrow \partial\Delta^1 \times E^\bullet \hookrightarrow \Delta^1 \times E^\bullet$$

(where the $\partial\Delta^1$ and Δ^1 denote constant cosimplicial spaces).

The map of cosimplicial spaces $\Delta^\bullet \to E^\bullet$ gives us a map $C_{cyl}^\bullet \to C_E^\bullet$, so that we have maps

(8.3)
$$\begin{tikzcd}
C_R^\bullet \arrow[dr] & & \\
& C_{cyl}^\bullet \arrow[r] & C_E^\bullet. \\
C_L^\bullet \arrow[ur] & &
\end{tikzcd}$$

Note that in each of these cosimplicial spaces, the 0th space is Δ^1. So every level of each of the above four cosimplicial spaces comes equipped with a canonical map to Δ^1.

Recall the notation $s\mathcal{S}et_{*,*} = (\partial\Delta^1 \downarrow s\mathcal{S}et_J)$.

Proposition 8.4. *(a) Each of the maps $c(\partial\Delta^1) \to C_R^\bullet$, $c(\partial\Delta^1) \to C_L^\bullet$, $c(\partial\Delta^1) \to C_{cyl}^\bullet$, and $c(\partial\Delta^1) \to C_E^\bullet$ is a Reedy cofibration.*

(b) Each of $C_R^\bullet$, $C_L^\bullet$, $C_{cyl}^\bullet$, and $C_E^\bullet$ is Reedy cofibrant as an object of $c(s\mathcal{S}et_{*,*})_J$.

(c) Each of the maps of simplicial sets $C_R^n \to \Delta^1$, $C_L^n \to \Delta^1$, $C_{cyl}^n \to \Delta^1$, and $C_E^n \to \Delta^1$ is a Joyal equivalence.

(d) Consequently, each of $C_R^\bullet$, $C_L^\bullet$, $C_{cyl}^\bullet$, and $C_E^\bullet$ is a cosimplicial resolution of Δ^1 with respect to the model category $(s\mathcal{S}et_{*,*})_J$ (where Δ^1 is regarded as an element of this undercategory in the usual way).

Proof. Parts (a) and (b) are obvious, and (d) follows immediately from (b) and (c). For (c), note first that $C_E^n \to \Delta^1$ is easily seen to be Joyal equivalence. For C_E^n is the pushout of

$$\partial\Delta^1 \xleftarrow{\sim} \partial\Delta^1 \times E^n \hookrightarrow \Delta^1 \times E^n,$$

where the indicated map is a Joyal equivalence by Lemma 8.2. It follows from left properness of $s\mathcal{S}et_J$ that $\Delta^1 \times E^n \to C_E^n$ is a Joyal equivalence. Using that $\Delta^1 \times E^n \to \Delta^1$ is a Joyal equivalence (Lemma 8.2 again), it follows immediately that $C_E^n \to \Delta^1$ is also one.

The arguments for C_R^n, C_L^n, and C_{cyl}^n are more complicated. Picking the former for concreteness, there are various sections of the map $C_R^n \to \Delta^1$. It will be sufficient to show that any one of these is an acyclic cofibration, which we do by exhibiting it as a composition of cobase changes of inner horn inclusions. This is not difficult, but it is a little cumbersome; we postpone the proof until the appendix, as Lemma A.7. □

8.5. **Application to mapping spaces.** For every $S \in s\mathcal{S}et_J$ and every $a, b \in S$, define $\mathrm{Hom}^R_S(a,b)$ to be the simplicial set $s\mathcal{S}et_{*,*}(C^\bullet_R, S)$. Note that this is also the pullback of

$$* \xrightarrow{(a,b)} S \times S \twoheadleftarrow s\mathcal{S}et(C^\bullet_R, S).$$

Define $\mathrm{Hom}^L_S(a,b)$, $\mathrm{Hom}^{cyl}_S(a,b)$, and $\mathrm{Hom}^E_S(a,b)$ analogously, and note that Diagram (8.3) induces natural maps

(8.6)
$$\begin{tikzcd} & & \mathrm{Hom}^R_S(a,b) \\ \mathrm{Hom}^E_S(a,b) \arrow[r] & \mathrm{Hom}^{cyl}_S(a,b) \arrow[ur] \arrow[dr] & \\ & & \mathrm{Hom}^L_S(a,b). \end{tikzcd}$$

Corollary 8.7. *When $S \in s\mathcal{S}et_J$ is fibrant and $a, b \in S$, the four natural maps in (8.6) are Kan equivalences of simplicial sets. These simplicial sets are models for the homotopy function complex $\mathrm{hMap}_{s\mathcal{S}et_{*,*}}(\Delta^1, S)$, where S is regarded as an object of $s\mathcal{S}et_{*,*}$ via the map $\partial\Delta^1 \to S$ sending $0 \to a$ and $1 \to b$.*

Proof. This is immediate from Proposition 8.4 and [H, 16.5.5]. □

Remark 8.8. For a simplicial set S and vertices $a, b \in S_0$, our notation $\mathrm{Hom}^R_S(a,b)$ and $\mathrm{Hom}^L_S(a,b)$ agrees with that of [L, Section 1.2.2]. Our notation $\mathrm{Hom}^{cyl}_S(a,b)$ is what Lurie denotes $\mathrm{Hom}_S(a,b)$. It can also be described as the fiber of the morphism of simplicial mapping spaces $\mathrm{Map}_{s\mathcal{S}et}(\Delta^1, S) \to \mathrm{Map}_{s\mathcal{S}et}(\partial\Delta^1, S)$ over the point (a,b). The model $\mathrm{Hom}^E_S(a,b)$ does not seem to appear in [L].

The following calculation will be needed in the next section:

Proposition 8.9. *Let T be a necklace. Then $\mathrm{hMap}_{(s\mathcal{S}et_{*,*})_J}(\Delta^1, T)$ is contractible.*

Proof. Recall from Lemma 3.1 that $T \to \Delta[T]$ is a weak equivalence in $s\mathcal{S}et_J$. Also, $\Delta[T]$ is fibrant in $s\mathcal{S}et_J$ because it is the nerve of a category (as is any Δ^k). We may therefore model our homotopy function complex by

$$s\mathcal{S}et_{*,*}(C^\bullet_R, \Delta[T])$$

where $C^\bullet_R$ is the cosimplicial resolution of Δ^1 considered in this section.

It is easy to check that in $s\mathcal{S}et_{*,*}$ there is a unique map from C^n_R to $\Delta[T]$, for each n. Therefore we have $s\mathcal{S}et_{*,*}(C^\bullet_R, \Delta[T]) = *$, and this completes the proof. □

9. Connections between the two approaches

In this section we prove that for any simplicial set S and any $a, b \in S_0$, the categorification mapping space $\mathfrak{C}(S)(a,b)$ is naturally weakly equivalent to the Dwyer-Kan mapping space $\mathrm{hMap}_{(s\mathcal{S}et_{*,*})_J}(\Delta^1, S)$. As a corollary, we prove that for any simplicial category $\mathcal{D}$ the map $\mathfrak{C}(N\mathcal{D}) \to \mathcal{D}$ is a weak equivalence in $s\mathcal{C}at$.

If S is a simplicial set and $a, b \in S$, let us use the notation $\mathrm{hMap}(S)_{a,b}$ as shorthand for a homotopy function complex $\mathrm{hMap}_{(s\mathcal{S}et_{*,*})_J}(\Delta^1, S)$.

Let $\mathcal{Y}$ denote the full subcategory of $s\mathcal{S}et_{*,*}$ whose objects are spaces Y such that $\mathrm{hMap}(Y)_{\alpha,\omega} \simeq *$ and $\mathfrak{C}(Y)(\alpha, \omega) \simeq *$. Note that $\mathcal{Y}$ contains the category $\mathcal{N}ec$, by Proposition 8.9 and Corollary 3.9. So clearly $\mathcal{Y}$ is a *category of gadgets* in the sense of Definition 5.4. Let $\mathcal{Y}_f$ denote the full subcategory of $\mathcal{Y}$ consisting of those objects which are fibrant in $s\mathcal{S}et_J$. Let $\mathfrak{C}^{\mathcal{Y}}$ and $\mathfrak{C}^{\mathcal{Y}_f}$ be as defined in Section 5.3.

Remark 9.1. Once we are done proving the main result of this section, we will know that the conditions $\mathfrak{C}(Y)(\alpha, \omega) \simeq *$ and $\mathrm{hMap}(Y)_{\alpha,\omega} \simeq *$ are equivalent. At the moment, however, we do not know this; so including both conditions in the definition of $\mathcal{Y}$ is not redundant.

Let $C^\bullet$ be a cosimplicial resolution of Δ^1 in $s\mathcal{S}et_J$. Let $C^\bullet \overset{\sim}{\rightarrowtail} R^\bullet \overset{\sim}{\twoheadrightarrow} c(\Delta^1)$ be a factorization into a Reedy acyclic cofibration followed by Reedy fibration (which will necessarily be acyclic as well). By [H, Prop. 15.3.11] the maps $R^n \to \Delta^1$ are fibrations in $s\mathcal{S}et_J$. Since Δ^1 is fibrant in $s\mathcal{S}et_J$ (being the nerve of a category), the objects R^n are fibrant as well.

Proposition 9.2. *If S is fibrant in $s\mathcal{S}et_J$ and $a, b \in S_0$, then the following is a commutative diagram in which all the maps are Kan equivalences:*

$$\begin{tikzcd}
\mathfrak{C}^{nec}(S)(a,b) \arrow[r, "\sim"] & \mathfrak{C}^{\mathcal{Y}}(S)(a,b) & \mathfrak{C}^{\mathcal{Y}_f}(S)(a,b) \arrow[l, "\sim"'] \\
& N\mathbf{\Delta}\,\mathrm{Hom}(C^\bullet, S_{a,b}) \arrow[u, "\sim"] & N\mathbf{\Delta}\,\mathrm{Hom}(R^\bullet, S_{a,b}). \arrow[l, "\sim"] \arrow[u, "\sim"']
\end{tikzcd}$$

Proof. Proposition 5.5 shows that $\mathfrak{C}^{nec}(S)(a,b) \to \mathfrak{C}^{\mathcal{Y}}(S)(a,b)$ is a Kan equivalence. Let us show that $\mathfrak{C}^{\mathcal{Y}_f}(S)(a,b) \to \mathfrak{C}^{\mathcal{Y}}(S)(a,b)$ is a Kan equivalence. Recall that this map is the nerve of the evident inclusion of categories $j \colon (\mathcal{Y}_f \downarrow S)_{a,b} \to (\mathcal{Y} \downarrow S)_{a,b}$. For a simplicial set X, let $X \overset{\sim}{\rightarrowtail} \hat{X}$ denote a (functorial) fibrant replacement of X in $s\mathcal{S}et_J$. Since S is fibrant, there is a map $\hat{S} \to S$ such that the composition $S \to \hat{S} \to S$ is the identity. Define a functor

$$F \colon (\mathcal{Y} \downarrow S)_{a,b} \to (\mathcal{Y}_f \downarrow S)_{a,b}$$

by sending the pair $[Y, Y \to S]$ to the pair $[\hat{Y}, \hat{Y} \to \hat{S} \to S]$. For this to make sense we need to know that $\hat{Y}$ is in $\mathcal{Y}_f$; this is true because changing from $Y \to \hat{Y}$ does not change the Dwyer-Kan mapping space $\mathrm{hMap}(-)_{a,b}$ nor, by Proposition 6.6, the $\mathfrak{C}(-)(a,b)$ mapping space. It is easy to see that there is a natural transformation between the composite jF (resp. Fj) and the identity, so j induces a Kan equivalence of the nerves.

Next consider the map $N\mathbf{\Delta}\,\mathrm{Hom}(R^\bullet, S_{a,b}) \to \mathfrak{C}^{\mathcal{Y}_f}(S)(a,b)$. This is again the nerve of a functor

$$f \colon \mathbf{\Delta}\,\mathrm{Hom}(R^\bullet, S_{a,b}) \to (\mathcal{Y}_f \downarrow S)_{a,b}$$

which sends $[[n], R^n \to S]$ to $[R^n, R^n \to S]$. We will verify that the overcategories of f are contractible, hence it induces a Kan equivalence of the nerves. (For typographical reasons, we may drop the subscripts a, b, etc.) Pick an object $y = [Y, Y \to S]$ in $(\mathcal{Y}_f \downarrow S)$. The overcategory $(f \downarrow y)$ has objects $[[n], R^n \to Y]$ and the evident morphisms; that is, $(f \downarrow y) = \mathbf{\Delta}\,\mathrm{Hom}(R^\bullet, Y)$. But since Y is fibrant, $\mathrm{Hom}(R^\bullet, Y)$ is a model for $\mathrm{hMap}(Y)_{a,b}$, and this is contractible because $Y \in \mathcal{Y}_f$.

The map $\operatorname{Hom}(R^\bullet, S) \to \operatorname{Hom}(C^\bullet, S)$ is a Kan equivalence because $C^\bullet \to R^\bullet$ is a Reedy weak equivalence between Reedy cofibrant objects and S is fibrant; see [H, 16.5.5]. Hence, the map $N\mathbf{\Delta}\operatorname{Hom}(R^\bullet, S_{a,b}) \to N\mathbf{\Delta}\operatorname{Hom}(C^\bullet, S_{a,b})$ is a Kan equivalence.

The final map $\mathbf{\Delta}\operatorname{Hom}(C^\bullet, S_{a,b}) \to \mathfrak{C}^{\mathcal{Y}}(S)(a,b)$ is a Kan equivalence by the two-out-of-three property. ∎

For the rest of this section we write $\mathcal{A} = s\mathcal{S}et_{*,*}$, to ease the cumbersome typography.

The above proposition gives a simple zig-zag of Kan equivalences between $\mathfrak{C}^{nec}(S)(a,b)$ and $N\mathbf{\Delta}\operatorname{Hom}(C^\bullet, S_{a,b})$ for any cosimplicial resolution $C^\bullet$ of Δ^1 in $s\mathcal{S}et_J$. In Proposition 9.3 we will present another simple zig-zag which is sometimes useful. Define

$$\phi\colon (\mathcal{N}ec \downarrow S)_{a,b} \to \mathcal{A}W^{-1}\mathcal{A}(\Delta^1, S_{a,b}) \tag{9.2}$$

by sending $[T, T \to S]$ to $[\Delta^1 \to \Delta[T] \xleftarrow{\sim} T \to S]$. Here $\Delta[T]$ is the associated simplex to T, described in Section 3, which is functorial in T. The map $\Delta^1 \to \Delta[T]$ is the unique 1-simplex connecting the initial and final objects. Note that there is also a functor

$$j\colon \mathbf{\Delta}\operatorname{Hom}(C^\bullet, S) \to \mathcal{A}W^{-1}\mathcal{A}(\Delta^1, S) \tag{9.2$'$}$$

which sends $[[n], C^n \to S]$ to $[\Delta^1 \xrightarrow{\mathrm{id}} \Delta^1 \xleftarrow{\sim} C^n \to S]$, and by Remark 7.5 this functor induces a Kan equivalence on nerves.

Proposition 9.3. *For any fibrant simplicial set $S \in s\mathcal{S}et_J$ and $a, b \in S_0$, the maps*

$$\mathfrak{C}^{nec}(S)(a,b) \xrightarrow{N\phi} N\big[\mathcal{A}W^{-1}\mathcal{A}(\Delta^1, S_{a,b})\big] \xleftarrow{Nj} N\big[\mathbf{\Delta}\operatorname{Hom}(C^\bullet, S_{a,b})\big],$$

where ϕ and j are as in (9.2) and (9.2$'$), are Kan equivalences.

Proof. We consider the following diagram of categories, where we have suppressed all mention of a and b, but everything is suitably over $\partial\Delta^1$:

$$\begin{tikzcd}
(\mathcal{N}ec \downarrow S) \arrow[rr] \arrow[d, "\phi"'] & & (\mathcal{Y} \downarrow S) & \mathbf{\Delta}\operatorname{Hom}(C^\bullet, S). \arrow[l] \arrow[dl, "j_0"] \\
\mathcal{A}W^{-1}\mathcal{A}(\Delta^1, S) \arrow[r, shift left, "\pi_2"] & \mathcal{A}\,\mathrm{Wfib}^{-1}\,\mathcal{A}(\Delta^1, S) \arrow[l, shift left, "j_2"] \arrow[r, shift left, "\pi_1"] & \mathcal{A}W^{-1}(\Delta^1, S) \arrow[l, shift left, "j_1"] \arrow[u, "i"'] &
\end{tikzcd}$$

The nerve of each map in the top row is from Proposition 9.2, where it is shown to be a Kan equivalence. The map ϕ was defined above. The maps j_0, j_1, j_2, i, π_1, and π_2 are in some sense self-evident, but we describe them now (in that order). The symbol " $\sim$ " in this proof always denotes a Joyal weak equivalence.

The map j_0 sends $[[n], C^n \to S]$ to $[\Delta^1 \xleftarrow{\sim} C^n \to S]$; j_1 sends $[\Delta^1 \xleftarrow{\sim} X \to S]$ to $[\Delta^1 \xrightarrow{\mathrm{id}} \Delta^1 \overset{\sim}{\twoheadleftarrow} X' \to S]$, where $X \overset{\sim}{\rightarrowtail} X' \twoheadrightarrow \Delta^1 \times S$ is a functorial factorization of $X \to \Delta^1 \times S$; and j_2 is induced by the inclusion $\mathrm{Wfib} \hookrightarrow W$. Note that the composite $j_2 j_1 j_0$ is the map j from the statement of the proposition. The map i sends $[\Delta^1 \xleftarrow{\sim} X \to S]$ to the pair $[X, X \to S]$ (note that if $X \simeq \Delta^1$ then $X \in \mathcal{Y}$ by Proposition 6.6 and the homotopy invariance of the Dwyer-Kan mapping spaces). Finally, the maps π_1 and π_2 are functors giving homotopy inverses to j_1 and j_2. The functor π_1 sends $[\Delta^1 \to X \overset{\sim}{\twoheadleftarrow} Y \to S]$ to $[\Delta^1 \xleftarrow{\sim} (\Delta^1 \times_X Y) \to S]$, and π_2 sends $[\Delta^1 \to X \xleftarrow{\sim} Y \to S]$ to $[\Delta^1 \to X \overset{\sim}{\twoheadleftarrow} Y' \to S]$ where Y' is obtained from the functorial factorization of $Y \to X \times S$ into $Y \overset{\sim}{\rightarrowtail} Y' \twoheadrightarrow X \times S$. It is easy to see that there are natural transformations between the composite $j_i\pi_i$, $\pi_i j_i$, and their respective identities, thus showing that these maps are homotopy inverses.

Next one should check that the functor $i\pi_1\pi_2\phi$ is connected to the top map $(\mathcal{N}ec \downarrow S) \to (\mathcal{Y} \downarrow S)$ by a zig-zag of natural transformations (this is easy), and hence the two maps induce homotopic maps on nerves. So the (nerve of the) large rectangle in the above diagram commutes in the homotopy category. The right-hand triangle commutes on the nose.

The map j_0 induces a Kan equivalence on nerves by Remark 7.5. Returning to our original diagram and the sentence immediately following it, the two-out-of-three property implies that i induces a Kan equivalence on nerves. We have already shown that $\pi_1\pi_2$ and j_2j_1 do so as well; therefore the same is true for ϕ and j. $\square$

Remark 9.4. The above result in some sense explains the role of necklaces in our story. If T is a necklace then a map $T \to S_{a,b}$ gives us, in a canonical way, a zig-zag

$$\Delta^1 \hookrightarrow \Delta[T] \overset{\sim}{\longleftarrow} T \to S$$

in $\mathcal{A}W^{-1}\mathcal{A}(\Delta^1, S)$, which represents a map $\Delta^1 \to S$ in $\mathrm{Ho}\,(s\mathcal{S}et_J)$.

9.5. **The counit of categorification.** Our next result concerns the counit $\epsilon\colon \mathfrak{C}N \to \mathrm{id}_{s\mathcal{C}at}$ for the adjunction $\mathfrak{C}\colon s\mathcal{S}et_J \rightleftarrows s\mathcal{C}at\colon N$. The proof is only a slight modification of that for Proposition 9.2 above. For a proof using very different methods, see [L, Theorem 2.2.0.1].

Proposition 9.6. *Let $\mathcal{D}$ be a simplicial category all of whose mapping spaces are Kan complexes. Then the counit map $\mathfrak{C}N\mathcal{D} \to \mathcal{D}$ is a weak equivalence in $s\mathcal{C}at$.*

Proof. Since $\mathfrak{C}(N\mathcal{D})$ is a simplicial category with the same object set as $\mathcal{D}$, it suffices to show that for any $a, b \in \mathrm{ob}\,\mathcal{D}$ the map

$$\mathfrak{C}(N\mathcal{D})(a,b) \to \mathcal{D}(a,b)$$

is a Kan equivalence.

Let $C^\bullet$ be the cosimplicial resolution $C^\bullet_R$ from Section 8, so that we have $C^n = (\Delta^n \star \Delta^0)/\Delta^n$. Observe that $\mathfrak{C}(C^n)$ is a simplicial category with two objects 0 and 1, and following [L, Section 2.2.2] let Q^n denote the mapping space $\mathfrak{C}(C^n)(0,1)$. By Proposition 6.6 and Proposition 8.4(c) the map $Q_n \to \mathfrak{C}(\Delta^1)(0,1) = *$ is a Kan equivalence, hence Q^n is contractible. Also, since $C^\bullet_R$ is Reedy cofibrant it follows readily that $Q^\bullet$ is also Reedy cofibrant. So the cosimplicial space $Q^\bullet$ is a Reedy cosimplicial resolution of a point in $s\mathcal{S}et_K$.

Consider the following diagram in $s\mathcal{S}et_K$:

$$\begin{tikzcd}
\mathfrak{C}(N\mathcal{D})(a,b) \arrow[r] & \mathcal{D}(a,b) & \\
\operatorname*{colim}_{T\to N\mathcal{D}} \mathfrak{C}(T)(\alpha,\omega) \arrow[u] \arrow[r] & \operatorname*{colim}_{Y\to N\mathcal{D}} \mathfrak{C}(Y)(\alpha,\omega) \arrow[u] & \operatorname*{colim}_{[n],C^n\to N\mathcal{D}} \mathfrak{C}(C^n)(\alpha,\omega) \arrow[l] \\
\operatorname*{hocolim}_{T\to N\mathcal{D}} \mathfrak{C}(T)(\alpha,\omega) \arrow[u] \arrow[r] \arrow[d, "\sim"'] & \operatorname*{hocolim}_{Y\to N\mathcal{D}} \mathfrak{C}(Y)(\alpha,\omega) \arrow[u] \arrow[d, "\sim"'] & \operatorname*{hocolim}_{[n],C^n\to N\mathcal{D}} \mathfrak{C}(C^n)(\alpha,\omega) \arrow[l] \arrow[u] \arrow[d, "\sim"'] \\
\operatorname*{hocolim}_{T\to N\mathcal{D}} * \arrow[r] & \operatorname*{hocolim}_{Y\to N\mathcal{D}} * & \operatorname*{hocolim}_{[n],C^n\to N\mathcal{D}} * \arrow[l]
\end{tikzcd}$$

For the colimits in the left-hand column the indexing category is $(\mathcal{N}ec \downarrow N\mathcal{D})_{a,b}$. For the middle column it is $(\mathcal{Y} \downarrow N\mathcal{D})_{a,b}$, where $\mathcal{Y}$ is the category of gadgets described at the beginning of this section. For the right-hand column the colimits are indexed by the category $\mathbf{\Delta}\operatorname{Hom}(C^\bullet, N\mathcal{D}_{a,b})$. The maps between columns (except at the very top) come from the evident maps between indexing categories. Finally, the top vertical map in the middle column comes from taking a map $Y \to N\mathcal{D}$, adjointing it to give $\mathfrak{C}(Y) \to \mathcal{D}$, and then using the induced map $\mathfrak{C}(Y)(\alpha,\omega) \to \mathcal{D}(a,b)$. It is easy to see that the diagram commutes.

The indicated maps are Kan equivalences because the mapping spaces in $\mathfrak{C}(T)$, $\mathfrak{C}(Y)$ and $\mathfrak{C}(C^n)$ are all contractible. The bottom horizontal row is $\mathfrak{C}^{nec}(N\mathcal{D})(a,b) \to \mathfrak{C}^{\mathcal{Y}}(N\mathcal{D})(a,b) \leftarrow N\mathbf{\Delta}\operatorname{Hom}(C^\bullet, N\mathcal{D}_{a,b})$, and these maps are Kan equivalences by Proposition 9.2. It follows that the horizontal maps in the third row are all Kan equivalences as well.

Now, the map $\operatorname{hocolim}_{[n],C^n\to N\mathcal{D}} \mathfrak{C}(C^n)(\alpha,\omega) \to \mathcal{D}(a,b)$ can be written as

$$\operatorname*{hocolim}_{[n],\mathfrak{C}(C^n)\to\mathcal{D}} \mathfrak{C}(C^n)(\alpha,\omega) \to \operatorname*{colim}_{[n],\mathfrak{C}(C^n)\to\mathcal{D}} \mathfrak{C}(C^n)(\alpha,\omega) \to \mathcal{D}(a,b).$$

To give a map $\mathfrak{C}(C^n) \to \mathcal{D}$ over a,b is exactly the same as giving a map $Q^n = \mathfrak{C}(C^n)(\alpha,\omega) \to \mathcal{D}(a,b)$. So the above maps may also be written as

$$\operatorname*{hocolim}_{[n],Q^n\to\mathcal{D}(a,b)} Q^n \longrightarrow \operatorname*{colim}_{[n],Q^n\to\mathcal{D}(a,b)} Q^n \longrightarrow \mathcal{D}(a,b).$$

By Lemma 9.7 below (using that $\mathcal{D}(a,b)$ is a Kan complex), the composite is a Kan equivalence.

It now follows from our big diagram that $\operatorname{hocolim}_{Y\to N\mathcal{D}} \mathfrak{C}(Y)(\alpha,\omega) \to \mathcal{D}(a,b)$ is a Kan equivalence. Finally, by Theorem 5.2 the map

$$\operatorname*{hocolim}_{T\to N\mathcal{D}} \mathfrak{C}(T)(\alpha,\omega) \to \mathfrak{C}(N\mathcal{D})(a,b)$$

is a Kan equivalence (this is the map $\mathfrak{C}^{hoc}(N\mathcal{D})(a,b) \to \mathfrak{C}(N\mathcal{D})(a,b)$ from the statement of that theorem). It now follows at once that $\mathfrak{C}(N\mathcal{D})(a,b) \to \mathcal{D}(a,b)$ is a Kan equivalence. □

Lemma 9.7. *Let $U^\bullet$ be any cosimplicial resolution of a point with respect to $s\mathcal{S}et_K$. Then for any Kan complex X, the composite*

$$\operatorname*{hocolim}_{[n],U^n\to X} U^n \longrightarrow \operatorname*{colim}_{[n],U^n\to X} U^n \longrightarrow X.$$

is a Kan equivalence.

Proof. The result is true for the cosimplicial resolution $\Delta^\bullet$ by [D2, Prop. 19.4]. There is a zig-zag $U^\bullet \to V^\bullet \leftarrow \Delta^\bullet$ of Reedy weak equivalences, where $V^\bullet$ is a fibrant replacement of $\Delta^\bullet$. Because of this, it is sufficient to show that if $U^\bullet \to V^\bullet$ is a map between cosimplicial resolutions of a point and we know the result for one of them, then we also know it for the other.

Let $I = \Delta\mathcal{M}(U^\bullet, X)$ and $J = \Delta\mathcal{M}(V^\bullet, X)$, and observe that our map $U^\bullet \to V^\bullet$ induces a functor $f\colon J \to I$.

Let $\Gamma_U\colon I \to \mathcal{M}$ be the functor $[[n], U^n \to X] \mapsto U^n$ and let $\Gamma_V\colon J \to \mathcal{M}$ be the functor $[[n], V^n \to X] \mapsto V^n$. Finally, let $\Theta\colon J \to \mathcal{M}$ be the functor $[[n], V^n \mapsto X] \mapsto U^n$. Note that there is a natural transformation $\Theta \to \Gamma_V$, and also a natural transformation $\Theta \to \Gamma_U \circ f$.

One considers the following diagram:

$$\begin{tikzcd}
N\Delta\mathcal{M}(U^\bullet, X) \arrow[r, equal] & \operatorname{hocolim}_I * & \operatorname{hocolim}_I \Gamma_U \arrow[l, "\sim"'] \arrow[r] & \operatorname{colim}_I \Gamma_U \arrow[dr] & \\
N\Delta\mathcal{M}(V^\bullet, X) \arrow[u] \arrow[r, equal] & \operatorname{hocolim}_J * \arrow[u] & \operatorname{hocolim}_J \Theta \arrow[l, "\sim"'] \arrow[u] \arrow[r] \arrow[d, "\sim"] & \operatorname{colim}_J \Theta \arrow[u] \arrow[r] \arrow[d] & X. \\
 & & \operatorname{hocolim}_J \Gamma_V \arrow[r] & \operatorname{colim}_J \Gamma_V \arrow[ur] &
\end{tikzcd}$$

The maps labelled $\sim$ are Kan equivalences because all the values of Γ_V, Γ_U, and Θ are contractible.

The key observation is that the map $\mathcal{M}(V^\bullet, X) \to \mathcal{M}(U^\bullet, X)$ is a Kan equivalence by [H, 16.5.5], since both $V^\bullet$ and $U^\bullet$ are cosimplicial resolutions of a point in $s\mathcal{S}et_K$ and X is fibrant. It follows that $N\Delta\mathcal{M}(V^\bullet, X) \to N\Delta\mathcal{M}(U^\bullet, X)$ is also a Kan equivalence, and applying the two-out-of-three axiom to the diagram we obtain that $\operatorname{hocolim}_I \Gamma_U \to X$ is a Kan equivalence if and only if $\operatorname{hocolim}_J \Gamma_V \to X$ is a Kan equivalence. This is what we wanted. □

9.8. **Joyal equivalences and categorification.** In this final section of the paper we use our previous results to establish the equivalence between the homotopy theories of quasi-categories and simplicial categories. This result was originally due to Lurie [L], and the proof at this point is essentially the same as his—the key step was Proposition 9.6 above.

Proposition 9.9. *A map of simplicial sets $X \to Y$ is a Joyal equivalence if and only if $\mathfrak{C}(X) \to \mathfrak{C}(Y)$ is a weak equivalence of simplicial categories.*

Proof. ????? □

Corollary 9.10. *The adjoint functors $\mathfrak{C}\colon s\mathcal{S}et_J \rightleftarrows s\mathcal{C}at\colon N$ are a Quillen equivalence.*

Proof. ????? □

Appendix A. Leftover proofs

In this section we give the proofs which were postponed in the body of the paper.

A.1. **Products of necklaces.** Our first goal is to prove Proposition 6.1. Let $T_1, \ldots, T_n$ be necklaces, and consider the product $X = T_1 \times \cdots \times T_n$. The main thing we need to prove is that whenever $a \preceq_X b$ in X the mapping space $\mathfrak{C}(X)(a,b) \simeq *$ is contractible.

Definition A.2. *An ordered simplicial set $(X, \preceq)$ is called **strongly ordered** if, for all $a \preceq b$ in X, the mapping space $\mathfrak{C}(X)(a,b)$ is contractible.*

Note that in any ordered simplicial set X with $a, b \in X_0$, we have $a \preceq b$ if and only if $\mathfrak{C}(X)(a,b) \neq \emptyset$. Thus if X is strongly ordered then its structure as a simplicial category, up to categorical equivalence, is completely determined by the ordering on its vertices. We also point out that every necklace $T \in \mathcal{N}ec$ is strongly ordered by Corollary 3.9.

Lemma A.3. *Suppose given a diagram*

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

where X, Y, and A are strongly ordered simplicial sets and both f and g are simple inclusions. Let $B = X \amalg_A Y$ and assume the following conditions hold:

(1) A has finitely many vertices;
(2) Given any $x \in X$, the set $A_{x\preceq} = \{a \in A \mid x \preceq_B a\}$ has an initial element (an element which is smaller than every other element).
(3) For any $y \in Y$ and $a \in A$, if $y \preceq_B a$ then $y \in A$.

Then B is strongly ordered.

Proof. By Lemma 3.6(8), B is an ordered simplicial set and the maps $X \hookrightarrow B$ and $Y \hookrightarrow B$ are simple inclusions. We must show that for $u, v \in B_0$ with $u \preceq v$, the mapping space $\mathfrak{C}(B)(u,v)$ is contractible. Suppose that u and v are both in X; then since $X \hookrightarrow B$ is simple, any necklace $T \to B_{u,v}$ must factor through X. It follows that $\mathfrak{C}(B)(u,v) = \mathfrak{C}(X)(u,v)$, which is contractible since X is strongly ordered. The case $u, v \in Y$ is analogous. We claim we cannot have $u \in Y \backslash A$ and $v \in X \backslash A$. For if this is so and if $T \to B$ is a spine connecting u to v, then there is a last vertex j of T that maps into Y. The 1-simplex leaving that vertex then cannot belong entirely to Y, hence it belongs entirely to X. So j is in both X and Y, and hence it is in A. Then we have $u \preceq j$ and $j \in A$, which by assumption (3) implies $u \in A$, a contradiction.

The only remaining case to analyze is when $u \in X$ and $v \in Y \backslash A$. Consider the poset A_0 of vertices of A, under the relation $\preceq$. Let P denote the collection of linearly ordered subsets S of A_0 having the property that $u \preceq a \preceq v$ for all $a \in S$. That is, each element of P is a chain $u \preceq a_1 \preceq \cdots \preceq a_n \preceq v$ where each $a_i \in A$. We regard P as a category, where the maps are inclusions. Also let P_0 denote the subcategory of P consisting of all subsets except $\emptyset$.

Define a functor $D\colon P^{op} \to \mathcal{C}at$ by sending $S \in P$ to

$$\{[T, T \hookrightarrow B_{u,v}] \ \mid \ S \subseteq J_T\},$$

the full subcategory of $(\mathcal{N}ec \downarrow B)_{u,v}$ spanned by objects $T \xrightarrow{m} B_{u,v}$ for which m is an injection and $S \subseteq J_T$. Let us adopt the notation

$$M_S(u,v) = \operatorname*{colim}_{T \in D(S)} \mathfrak{C}(T)(\alpha, \omega).$$

Note that there is a natural map

$$M_\emptyset(u,v) \longrightarrow \operatorname*{colim}_{T\in(\mathcal{N}ec\downarrow S)_{u,v}} \mathfrak{C}(T)(\alpha,\omega) \cong \mathfrak{C}(B)(u,v).$$

The first map is not *a priori* an isomorphism because in the definition of $D(\emptyset)$ we require that the map $T \to B$ be an injection. However, using Lemma 4.12 (or Corollary 4.13) it follows at once that the map actually is an isomorphism.

We claim that for each S in P_0 the "latching" map

$$L_S\colon \operatorname*{colim}_{S'\supset S} M_{S'}(u,v) \to M_S(u,v)$$

is an injection, where the colimit is over sets $S' \in P$ which strictly contain S. To see this, suppose that one has a triple $[T, T \hookrightarrow B_{u,v}, t \in \mathfrak{C}(T)(\alpha,\omega)_n]$ giving an n-simplex of $M_{S'}(u,v)$ and another triple $[T', T' \hookrightarrow B_{u,v}, t' \in \mathfrak{C}(U)(\alpha,\omega)_n]$ giving an n-simplex of $M_{S''}(u,v)$. If these become identical in $M_S(u,v)$ then it must be that they have the same flankification $\bar{T} = \bar{U}$ and $t = t'$. Note that every joint of T is a joint of $\bar{T}$, so the joints of $\bar{T}$ include both S' and S''. Because the joints of any necklace are linearly ordered, it follows that $S' \cup S''$ is linearly ordered. Since $T \to \bar{T}$ is an injection, we may consider the triple $[\bar{T}, \bar{T} \hookrightarrow B_{u,v}, t]$ as an n-simplex in $M_{S'\cup S''}(u,v)$, which maps to the two original triples in the colimit; this proves injectivity.

We claim that the latching map $L_\emptyset\colon \operatorname{colim}_{S\in P_0^{op}} M_S(u,v) \to M_\emptyset(u,v)$ is an isomorphism. Injectivity was established above. For surjectivity, one needs to prove that if T is a necklace and $T \hookrightarrow B_{u,v}$ is an inclusion, then T must contain at least one vertex of A as a joint. To see this, recall that every simplex of B either lies entirely in X or entirely in Y. Since $v \notin X$, there is a last joint j_1 of T which maps into X. If C denotes the bead whose initial vertex is j_1, then the image of C can not lie entirely in X; so it lies entirely in Y, which means that j_1 belongs to both X and Y—hence it belongs to A.

From here the argument proceeds as follows. We will show:

(i) The natural map $\operatorname{hocolim}_{S\in P_0^{op}} M_S(u,v) \to \operatorname{colim}_{S\in P_0^{op}} M_S(u,v)$ is a Kan equivalence;

(ii) Each $M_S(u,v)$ is contractible, hence the above homotopy colimit is Kan equivalent to the nerve ov P_0^{op};

(iii) The nerve of P_0 (and hence also P_0^{op}) is contractible.

This will prove that $M_\emptyset(u,v) = \mathfrak{C}(B)(u,v)$ is contractible, as desired.

For (i) we refer to [D2, Section 13] and use the fact that P_0^{op} has the structure of a directed Reedy category. Indeed, we can assign a degree function to P that sends a set $S \subseteq A_0$ to the nonnegative integer $|A_0 - S|$; all non-identity morphisms in P_0^{op} strictly increase this degree. By Proposition [D2, 13.3] it is enough to show that all the latching maps L_S are cofibrations, and this has already been established above.

For claim (iii), write θ for the initial vertex of $A_{u\preceq}$. Define a functor $F\colon P_0 \to P_0$ by $F(S) = S \cup \{\theta\}$; note that $S \cup \{\theta\}$ will be linearly ordered, so this makes sense. Clearly there is a natural transformation from the identity functor to F, and also from the constant $\{\theta\}$ functor to F. It readily follows that the identity map on NP_0 is homotopic to a constant map, hence NP_0 is contractible.

Finally, for (ii) fix some $S \in P_0$ and let $u = a_0 \prec a_1 \prec \ldots \prec a_n \prec a_{n+1} = v$ denote the complete set of elements of $S \cup \{u,v\}$. A necklace $T \hookrightarrow B_{u,v}$ whose joints include the elements of S can be split along the joints, and thus uniquely written as the wedge of necklaces $T_i \hookrightarrow B_{a_i,a_{i+1}}$, one for each $0 \le i \le n$. Under this identification, one has

$$\mathfrak{C}(T)(\alpha,\omega) = \mathfrak{C}(T_0)(\alpha_0,\omega_0) \times \cdots \times \mathfrak{C}(T_n)(\alpha_n,\omega_n).$$

Thus $D(p)$ is isomorphic to the category

$$(\mathcal{N}ec \downarrow^m X)_{u,a_1} \times (\mathcal{N}ec \downarrow^m A)_{a_1,a_2} \times \cdots \times (\mathcal{N}ec \downarrow^m A)_{a_{n-1},a_n} \times (\mathcal{N}ec \downarrow^m Y)_{a_n,v},$$

where $(\mathcal{N}ec \downarrow^m X)_{s,t}$ denotes the category whose objects are $[T, T \to X_{s,t}]$ where the map $T \to X$ is a monomorphism.

Now, it is a general fact about colimits taken in the category of (simplicial) sets, that if M_i is a category and $F_i \colon M_i \to s\mathcal{S}et$ is a functor, for each $i \in \{1,\ldots,n\}$, then there is an isomorphism of simplicial sets

$$\operatorname*{colim}_{M_1\times\cdots\times M_n}(F_1 \times \cdots \times F_n) \xrightarrow{\cong} \left(\operatorname*{colim}_{M_1} F_1\right) \times \cdots \times \left(\operatorname*{colim}_{M_n} F_n\right). \tag{A.2.2}$$

Applying this in our case, we find that

$$M_S(u,v) \cong \mathfrak{C}(X)(u,a_1) \times \mathfrak{C}(A)(a_1,a_2) \times \cdots \times \mathfrak{C}(A)(a_{n-2},a_{n-1}) \times \mathfrak{C}(Y)(a_{n-1},v).$$

Note that this is always contractible, since X, A, and Y are strongly ordered. This proves (ii) and completes the argument. □

Proposition A.4. *Let $T_1,\ldots,T_m$ be necklaces. Then their product $P = T_1 \times \cdots \times T_m$ is a strongly ordered simplicial set.*

Proof. We begin with the case $P = \Delta^{n_1} \times \cdots \times \Delta^{n_m}$, where each necklace is a simplex, and show that P is strongly ordered. It is ordered by Lemma 3.6, so choose vertices $a, b \in P_0$ with $a \preceq b$. If T is a necklace, any map $T \to \Delta^j$ extends uniquely to a map $\Delta[T] \to \Delta^j$. It follows that any map $T \to P_{a,b}$ extends uniquely to $\Delta[T] \to P_{a,b}$. Consider the two functors

$$f, g \colon (\mathcal{N}ec \downarrow P)_{a,b} \to (\mathcal{N}ec \downarrow P)_{a,b}$$

where f sends $[T, T \to P]$ to $[\Delta[T], \Delta[T] \to P]$ and g is the constant functor sending everything to $[\Delta^1, x \colon \Delta^1 \to P]$ where x is the unique edge of P connecting a and b. Then clearly there are natural transformations $\mathrm{id} \to f$ and $g \to f$, showing that the three maps id, f, and g induce homotopic maps on the nerves. So the identity induces the null map, hence $\mathfrak{C}^{nec}(P)(a,b) = N((\mathcal{N}ec \downarrow P)_{a,b})$ is contractible. The result for P now follows by Theorem 5.2.

For the general case, assume by induction that we know the result for all products of necklaces in which at most $k-1$ of them are not equal to beads. The case $k = 1$ was handled by the previous paragraph. Consider a product

$$Y = T_1 \times \cdots \times T_k \times D$$

where each T_i is a necklace and D is a product of beads. Write $T_k = B_1 \vee B_2 \vee \cdots \vee B_r$ where each B_i is a bead, and let

$$P_j = (T_1 \times \cdots \times T_{k-1}) \times (B_1 \vee \cdots \vee B_j) \times D.$$

We know by induction that P_1 is strongly ordered, and we will prove by a second induction that the same is true for each P_j. So assume that P_j is strongly ordered for some $1 \le j < r$.

Let us denote $A = (T_1 \times \cdots \times T_{k-1}) \times \Delta^0 \times D$ and

$$Q = (T_1 \times \cdots \times T_{k-1}) \times B_{j+1} \times D.$$

Then we have $P_{j+1} = P_j \amalg_A Q$, and we know that P_j, A, and Q are strongly ordered. Note that the maps $A \to P_{j+1}$ and $A \to Q$ are simple inclusions: they are the products of $\Delta^0 \to B_j$ (resp. $\Delta^0 \to B_{j+1}$) with identity maps, and any inclusion $\Delta^0 \to \Delta^m$ is clearly simple. It is easy to check that hypothesis (1)–(3) of Lemma A.3 are satisfied, and so this finishes the proof. $\square$

Proof of Proposition 6.1. This follows immediately from Proposition A.4. $\square$

A.5. **The category $\mathfrak{C}(\Delta^n)$.** Our next goal is to give the proof of Lemma 2.6. Recall that this says there is an isomorphism

$$\mathfrak{C}(\Delta^n)(i,j) \to N(P_{i,j})$$

for $n \in \mathbb{N}$ and $0 \leq i, j \leq n$, where $P_{i,j}$ is the poset of subsets of $\{i, i+1, \ldots, j\}$ containing i and j.

Proof of Lemma 2.6. The result is obvious when $n = 0$, so we assume $n > 0$. Let $X = (FU)_\bullet([n])(i,j)$ and $Y = P_{i,j}$. For each $\ell \in \mathbb{N}$, we will provide an isomorphism $X_\ell \cong Y_\ell$, and these will be compatible with face and degeneracy maps.

One understands $X_0 = FU([n])(i,j)$ as the set of free compositions of sequences of morphisms in $[n]$ which start at i and end at j. By keeping track of the set of objects involved in this chain, we identify X_0 with the set of subsets of $\{i, i+1, \ldots, j\}$ which contain i and j. This gives an isomorphism $X_0 \to Y_0$.

Similarly for $\ell > 0$, one has that X_ℓ is the set of free compositions of sequences of morphisms in $X_{\ell-1}$. This set is in one-to-one correspondence with the set of ways to "parenthesize" the sequence $i, \ldots, j$ in such a way that every element is contained in $(\ell+1)$-many parentheses (and no closed parenthesis directly follows an open parenthesis). Given such a parenthesized sequence, one can rank the parentheses by "interiority." The face and degeneracy maps on X are given by deleting or repeating all the parentheses of a fixed rank.

Under this description, a vertex in an ℓ-simplex in X is given by choosing a rank, and then taking all the parentheses of that rank. By looking at only the last elements before one such close-parenthesis, we get a subset of $\{i+1, \ldots, j\}$ containing j, and hence by unioning with $\{i\}$, a unique element of X_0. Given two ranks, the subset of $\{i+1, \ldots, j\}$ corresponding to the higher rank will contain the subset corresponding to the lower rank. In fact, since one sees immediately that an ℓ-simplex in X is determined by its set of vertices, we can identify X_ℓ with the set of sequences $a_0 \subset a_1 \subset \cdots \subset a_\ell \subset \{i, i+1, \ldots, j\}$ containing i and j. This is precisely the set of ℓ-simplices of Y, so we have our isomorphism. It is clearly compatible with face and degeneracy maps. $\square$

A.6. **Models for Δ^1 in $s\mathcal{S}et_J$.** Our final goal is to complete the proof of Proposition 8.4 by showing that for any $n \geq 1$ the canonical maps $C^n_R \to \Delta^1$, $C^n_L \to \Delta^1$, and $C^n_{cyl} \to \Delta^1$ are all Joyal equivalences.

Given integers $0 \leq k < n$, define Δ^n_k to be the quotient of Δ^n obtained by collapsing the initial Δ^k to a point and the terminal Δ^{n-k-1} to a (different) point. Note that Δ^n_k has exactly two vertices, and there is a unique surjection $\Delta^n_k \to \Delta^1$. Note also that $\Delta^n_{n-1} = C^n_R$ and $\Delta^n_0 = C^n_L$.

Lemma A.7. *For integers $0 \leq k < n$, the surjection $\Delta^n_k \to \Delta^1$ is a Joyal equivalence.*

Proof. When $n = 1$ the map is the identity and there is nothing to prove, so assume $n \geq 2$. The proof in the case $k = 0$ is different than all the others; we will treat it at the end. Assume $k \geq 1$.

Any non-empty ordered subset $S \subseteq \{0, \ldots, n\}$ represents a simplex in Δ^n_k, which we denote $\Delta^S \in s\mathcal{S}et_{/\Delta^n_k}$, and that simplex is non-degenerate if and only if $S \nsubseteq \{0, \ldots, k\}$ and $S \nsubseteq \{k+1, \ldots, n\}$. For $s \in S$ we will write $\Lambda^S_s \in s\mathcal{S}et_{/\Delta^n_k}$ to denote the horn in Δ^S with distinguished point s.

Let $X = X_1 = \Delta^{\{k,k+1\}}$ in $s\mathcal{S}et_{/\Delta^n_k}$. We will show that the (injective) map $X \to \Delta^n_k$ can be written as the composite of finitely many cobase extensions along inner horn inclusions. This will prove the result by two-out-of-three.

Let $S \subseteq \{0, \ldots, n\}$ be a set of cardinality 3, such that $k \in S \nsubseteq \{0, \ldots, k\}$. If k is not the smallest element of S, let $\ell = k$; if k is the smallest, let ℓ be the second element. Note that $\Delta^S \to \Delta^n_k$ is non-degenerate. There is a unique map from the inner horn $\Lambda^S_\ell \to X_1$ (over Δ^n_k), and the pushout $X^S_1 = \Delta^S \amalg_{\Lambda^S_\ell} X_1$ has three nice properties:

(1) $X \to X^S_1$ is a Joyal equivalence
(2) every non-degenerate 2-simplex in X^S_1 contains the vertex k, and
(3) the canonical map $X^S_1 \to \Delta^n_k$ is injective.

Choose another set $S' \neq S$ as above, and let $X^{S,S'}_1 = \Delta^{S'} \amalg_{\Lambda^{S'}_{\ell'}} X^S_1$ be the corresponding pushout. It again has the three nice properties. Continue this process with every set of cardinality 3, containing k and not contained in $\{0, \ldots, k\}$, and call the end result $X_2 \in s\mathcal{S}et_{/\Delta^n_k}$. The above properties hold (with X_2 written in place of X^S_1), and in fact property (2) can be strengthened:

(2′) a non-degenerate 2-simplex in Δ^n_k is in the image of X_2 if and only if it contains the vertex k.

If $n = 2$, we are done. If not, let $S \subseteq \{0, \ldots, n\}$ be a set of cardinality 4, such that $k \in S \nsubseteq \{0, \ldots, k\}$, let ℓ be as above, and again form $X^S_2 = \Delta^S \amalg_{\Lambda^S_\ell} X_2$; it has the same nice properties (with all instances of "2" replaced by "3" and all instances of "1" replaced by "2"). Again continue with all such S and call the final result X_3; note that it also satisfies the appropriate analogue of property (2'). If $n = 3$, we are done. If not, continue forming X_i with the desired properties until $n = i$.

We have proven the result in case $k \geq 1$. The above argument can be dualized (by reversing the order of all the ordered sets involved). Noting that Δ^n_0 is in this sense dual to Δ^n_{n-1}, one sees that the case $k = 0$ is argued dually to the case $k = n - 1$ proven above.

□

Proposition A.8. *For every $n \geq 0$, the maps $C^n_R \to \Delta^1$, $C^n_L \to \Delta^1$, and $C^n_{cyl} \to \Delta^1$ are Joyal equivalences.*

Proof. The cases of C^n_R and C^n_L follow immediately from Lemma A.7. The case of C^n_{cyl} also follows from that lemma by a simple calculation, which we show below.

Let $\{0, 1, \ldots, n\}$ and $\{0', 1', \ldots, n'\}$ denote the vertices in $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$, respectively; since $\Delta^n \times \Delta^1$ is an ordered simplicial set, any simplex in it can be represented by its sequence of vertices. There is a unique way to write $\Delta^n \times \Delta^1$ as the union of n-many non-degenerate $(n+1)$-simplices. For example if $n = 3$ we have:

$$\Delta^3 \times \Delta^1 = \Delta^{\{0,1,2,2'\}} \amalg_{\Delta^{\{0,1,2'\}}} \Delta^{\{0,1,1',2'\}} \amalg_{\Delta^{\{0,1',2'\}}} \Delta^{\{0,0',1',2'\}}$$

Recall that C^n_{cyl} is obtained from $\Delta^n \times \Delta^1$ by collapsing any simplex whose vertices are contained in $\{0, \ldots, n\}$ or in $\{0', \ldots, n'\}$. We can write it as the union

$$C^n_{cyl} \cong \Delta^n_{n-1} \amalg_{\Delta^{n-1}_{n-2}} \Delta^n_{n-2} \amalg_{\Delta^{n-1}_{n-3}} \cdots \amalg_{\Delta^{n-1}_0} \Delta^n_0.$$

Each of the above maps $\Delta^{n-1}_i \to \Delta^n_i$ and $\Delta^{n-1}_i \to \Delta^n_{i+1}$ are cofibrations. They are also weak equivalences by Lemma A.7 (and two-out-of-three). Hence this colimit is contractible, completing the proof.

□

Appendix B. The Box-product lemma

Lemma B.1. *For integers $0 < k < n$ and r, the map*

$$f = (\Lambda^n_k \to \Delta^n)\Box(\partial\Delta^r \to \Delta^r)$$

is inner anodyne.

Proof. Let $X = (\Lambda^n_k \times \Delta^r) \amalg_{\Lambda^n_k \times \partial\Delta^r} (\Delta^n \times \partial\Delta^r)$ and $Y = \Delta^n \times \Delta^r$. We want to show that $f\colon X \to Y$ can be written as the composition of pushouts of inner horn inclusions. For the remainder of this proof, all simplicial sets under consideration will denote objects of the over-category $s\mathcal{S}et_{/Y}$. Given an object $Z \to Y$, we may speak of simplices of Y being *in* Z, by which we mean that they are in the image.

Let us establish some notation. An m-simplex y in Y is determined by its vertices, and we can denote it in the form

$$y = \begin{pmatrix} a_0 & a_1 & \ldots & a_m \\ b_0 & b_1 & \ldots & b_m \end{pmatrix},$$

where $0 \le a_i \le a_{i+1} \le n$ and $0 \le b_i \le b_{i+1} \le r$, for $0 \le i < m$. It is degenerate if and only if there exists i such that $a_i = a_{i+1}$ and $b_i = b_{i+1}$. Note that no two non-degenerate simplices of Y share any horn. We denote the face obtained by removing the vertex $\binom{a_i}{b_i}$ from y by $d_{\binom{a_i}{b_i}}(y)$; note that if that vertex is repeated then it does not matter which copy we remove.

One checks that y is an element of X if and only if it satisfies one of the following two conditions:

(i) $k \in \{a_0, a_1, \ldots, a_m\} \neq \{0, 1, \ldots, n\}$, -OR-
(ii) $\{b_0, \ldots, b_m\} \neq \{0, 1, \ldots, r\}$

Our strategy of proof is this. For $0 \le j \le r$ and $n \le t \le n + r$, we will produce maps $X \to X^t_j$ in $s\mathcal{S}et_{/Y}$ with three properties:

(1) $X \to X^t_j$ is inner anodyne,
(2) $X^t_j \to Y$ is injective,
(3) for every t'-simplex y in Y containing a vertex $\binom{k}{j'}$, where $t' \le t$ and $j' \le j$, we have $y \in X^t_j$,
(4) every t-simplex of $X^t_j - X^{n+r}_{j-1}$ contains $\binom{k}{j}$. When $j = 0$ we set $X^{n+r}_{-1} = X$, and
(5) no simplex of $X^t_j - X$ contains a vertex $\binom{k'}{j+1}$ for $k' < k$.

Moreover, we will have $X^s_{j'} \subseteq X^t_j$ either if $j' < j$ or if $j' = j'$ and $s \leq t$. Properties (1), (2), and (3) are essential; Properties (4) and (5) are for bookkeeping. Since Y is the union of its non-degenerate $(n+r)$-simplices and every such simplex contains a vertex of the form $\binom{k}{j}$ for some $0 \leq j \leq r$, the proof will be complete once we have done the $j = r$, $t = n + r$ case.

If $g\colon Z \to Y$ is an injection then for any $0 < s < t$ and diagram

$$\begin{tikzcd} \Lambda^t_s \arrow[r, "y_s"] \arrow[d] & Z \arrow[d, "g"] \\ \Delta^t \arrow[r, "y"'] & Y, \end{tikzcd}$$

the induced map $Z \amalg_{\Lambda^t_s} \Delta^t \to Y$ will be an injection if and only if $\Lambda^t_s \to Z$ does not extend to a map $\partial\Delta^t \to Z$ (over y).

Begin with $j = 0$, and let

$$y = \begin{pmatrix} a_0 & \dots & k & \dots & a_t \\ b_0 & \dots & 0 & \dots & b_t \end{pmatrix} \tag{B.1}$$

denote an arbitrary non-degenerate t-simplex in $Y - X$ that contains $\binom{k}{0}$. We will construct X^t_0 by induction on $t \geq n$.

Suppose $t = n$ and take y as above. Note that since y is non-degenerate, $d_{\binom{a_i}{b_i}} = d_i$. For all $s \neq k$, we have the face $d_s(y) \in X$ by (i), and hence a map $y_k\colon \Lambda^n_k \to X$ over y. One also checks that $d_k(y) \notin X$, so if we let

$$X^y = \operatorname{colim}(X \xleftarrow{y_k} \Lambda^n_k \to \Delta^n)$$

then $X \to X^y$ is inner anodyne and the induced map $X^y \to Y$ is injective. Note that X^y has one new n-simplex, y, and one new $(n-1)$-simplex, $d_k(y)$.

Now choose different n-simplex $y \neq y' \in Y - X$, with $\binom{k}{0} \in y'$. Again, one shows that there is a map $y'_k\colon \Lambda^n_k \to X^y$ over y'. Note that the face $d_k(y')$ cannot be in X^y because one can check that it is neither in X nor in y (if it were in y then we would have $\partial y = \partial y'$, hence $y = y'$). Let $X^{\{y,y'\}} = \operatorname{colim}(X^y \xleftarrow{y'_k} \Lambda^n_k \to \Delta)$ as above. Proceeding in this way, we add all the n-simplices of $Y - X$ which contain $\binom{k}{0}$, and the final result X^n_0 will satisfy the five properties.

Suppose X^{t-1}_0 has been constructed for some $t > n$, and let $y \in Y - X^{t-1}_0$ denote a non-degenerate t-simplex that contains $\binom{k}{0}$ as in (B.1). By induction and Property (3), there is a unique map $y_k\colon \Lambda^t_k \to X^t_0$ over y; we must check that it does not extend over $d_{\binom{k}{0}}(y)$. Beginning with the fact that $y \notin X$, one can argue that $d_{\binom{k}{0}}(y) \notin X$. Property (4) then ensures that $d_{\binom{k}{0}}(y) \notin X^{t-1}_0$. As above, let $X^y_0 = \operatorname{colim}(X \xleftarrow{y_k} \Lambda^t_k \to \Delta^n)$. Continue to add all the t-simplices of $Y - X^{t-1}_0$ which contain $\binom{k}{0}$, and the final result X^t_0 will satisfy the five properties. We have completed the $j = 0$ case.

Suppose $X_j := X^{n+r}_j$ has been constructed, for some $j \geq 0$, satisfying the five properties. Let

$$y = \begin{pmatrix} a_0 & \dots & k & \dots & a_t \\ b_0 & \dots & j+1 & \dots & b_t \end{pmatrix}$$

denote an arbitrary non-degenerate t-simplex in $Y - X_j$ containing $\binom{k}{j+1}$.

Note that since $y \notin X_j$, it contains no vertex of the form $\binom{k}{j'}$ for $j' \leq j$, by Property (3. Also, y cannot be of the form $\begin{pmatrix} a_0 & \dots & k' & k & \dots & a_t \\ b_0 & \dots & j' & j+1 & \dots & b_t \end{pmatrix}$, for $k' < k$ and $j' \leq j$, because if it were then it would be a face of the simplex $\begin{pmatrix} a_0 & \dots & k' & k & k & \dots & a_t \\ b_0 & \dots & j' & j' & j+1 & \dots & b_t \end{pmatrix}$, which is in X_j by (3). In other words, an arbitrary t-simplex in $Y - X_j$ must be of the form

$$y = \begin{pmatrix} a_0 & \dots & k' & k & \dots & a_t \\ b_0 & \dots & j+1 & j+1 & \dots & b_t \end{pmatrix} \tag{B.2}$$

for some $k' < k$. We will construct each X_{j+1}^t by induction on $t \geq n$.

Suppose $t = n$; as usual, it is easy to see that there is a unique map $\Lambda_k^n \to X_j$ over y and our job is to show that it does not extend to $\partial\Delta^n$. In other words, we must show that the face $d_{\binom{k}{j+1}}(y)$ is not in X_j. One argues that it is not in X, and it is not in $X_j - X$ by Property (5), so it is not in X_j. Continue to add all the n-simplices of $Y - X_j$ containing $\binom{k}{j+1}$, and the final result X_{j+1}^n will satisfy the five properties.

Suppose X_{j+1}^{t-1} has been constructed for some $t > n$, and let $y \in Y - X_{j+1}^{t-1} \subseteq Y - X_j$ be a t-simplex that contains $\binom{k}{j+1}$ as in (B.2). By induction and Property (3), there is a unique map $\Lambda_k^t \to X_{j+1}^{t-1}$ over y and it only remains to show that $d_{\binom{k}{j+1}}(y)$ is not in X_{j+1}^{t-1}. Again, one argues that it is not in X, and it is not in X_{j+1}^{t-1} by Property (5), so we are done.

With the induction now complete, the result is that we can find $X \to X_r^{n+r} \to Y$ satisfying properties (1) - (5). The point is to realize that $X_r^{n+r} \to Y$ is an isomorphism. It is injective by Property (2), and it is surjective by Property (3), since Y is the union of its $(n+r)$-simplices and each such simplex contains a vertex of the form $\binom{k}{s}$ for some $s \leq r$.

$\square$

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Department of Mathematics, University of Oregon, Eugene, OR 97403

Department of Mathematics, University of Oregon, Eugene, OR 97403
E-mail address: `ddugger@uoregon.edu`
E-mail address: `dspivak@uoregon.edu`