(Supplement to Functorial Dynamics and Interaction)

Data-driven dynamics and collective decision-making

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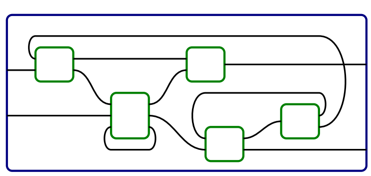
Statement of work

Original research scope

In the original project, AFOSR grant FA9550-20-1-0348, we proposed a mathematical formalism for modeling, designing, and simulating very general sorts of interactions [Spi20]. The mathematical formalism for this application is well-known and quite elegant; in category theory circles it's known as *polynomial functor theory*, and it has been studied by many eminent mathematicians and computer scientists [GH03; AAG03; AAG05; Koc+10; GK12; ACU14; AU16; Str19]. The latter have emphasized its power in programming languages and in dynamical systems [Has+09; PY15; Jac17]. However, no attention had formerly been paid to the ways the same theory could be used to model *interactions* between the resulting systems. This is what we proposed to study.

In a bit more detail, our interest is in understanding how the interactions of small systems can produce behavior of bigger systems. This includes proteins in a cell, cells in a body, or bodies in a society. Our previous work along these lines [Spi13; RS13; VSL15; Yau15; PSV21] emphasized fixed interaction patterns, as described by a wiring diagram:



Here each interior box (green) represents a system taking inputs (wires coming in on its left) and producing outputs (wiring leaving on its right). The exterior box (blue) could be seen as a new larger system—again having inputs and outputs—which interacts with the wider world according to the information-shuttling organization that is dictated by the wiring diagram. The wiring diagram represents a *fixed interaction pattern* between the parts.

But in the real world, while nested systems of fixed interaction patterns are found in computers and other manufactured objects (e.g. transistors wired to make logic gates, which are wired together to make adder circuits, then cores, then computers, then server farms, etc), living systems tend to have *variable interaction patterns*. For example, cells move around, people change who they talk to, organizations are formed and have turnover, empires collapse, etc. Thus in [ST17], we invented a notion of mode-dependent wiring diagrams: interaction patterns that can vary based on the internal states of the systems involved. The problem was that even though we were able to model variable interaction, the math was not elegant.

The current proposal started when we noticed that the elegant mathematics discussed above (polynomial functor theory) quite naturally fits the notion of mode-dependent systems. We set out to develop that story mathematically, with plenty of examples, and put it into working code.

Summary of supplemental work

Since the project began in September 2020, we have made substantial progress in developing the mathematical formalism and making intuitive sense of the theorems we proved about it. In particular, we have seen that deep learning (gradient descent and backpropagation) [LBH15; FST19] in fact falls within this paradigm [Spi21]. We have also formulated a mathematical structure called a *collective* that serves as a kind of economic system: a protocol for aggregating economic contributions, and a protocol of distributing returns or rewards to the contributors, based on their contributions. Finally, we've seen how to generalize the mode-dependent dynamical systems into systems that have an ontology—a schematic way of conceptualizing the world—and that interact with the world in reference to that ontology.

Whereas we would spend some amount of time on these topics anyway without supplemental funding, these topics would not be central. For this supplemental work, we propose to further develop the above three mathematical stories as the principal focus.

Details on supplemental work

The three projects we are aiming at in this supplemental work are:

- to further consider learning organizations, a generalization of deep learning;
- to develop the theory of collectives, a sort of economic system that falls out of the polynomial functor theory; and
- to understand the dynamics of systems that interact in the world via an internal ontology.

We briefly discuss each of these in turn.

Learning organizations In [Spi21], we explained that the very same dynamics that allows agents to change their wiring diagram based on internal states is also found in the operation of deep learning systems. Namely, for sets A and B, a machine that takes in $(a, b) \in A \times B$ training pairs and updates some parameters (weights and biases) can be understood in the language of polynomial functors as a $[Ay^A, By^B]$-coalgebra, where $[-,-]$ represents the closure of the $(y, \otimes)$ monoidal structure on $\mathbf{Poly}$. The gradient descent and backpropagation algorithm turns out to be a certain logical proposition in the topos $\mathcal{E} := [Ay^A, By^B]$–$\mathbf{Coalg}$ of such coalgebras. But much more is possible there: we can consider other algorithms in the internal language of this topos and then use the dependent type theory and higher-order logic of $\mathcal{E}$ to model much more interesting learning behaviors. What's being *learned* is how to adjust the wiring diagram itself, based on what flows through the wires.

Collectives As mentioned above, the category of polynomial functors has a monoidal structure $(\mathbf{Poly}, y, \otimes)$, called the Dirichlet product. It is used in combining systems in a fixed wiring diagram or variable interaction pattern: given $p_1, \ldots, p_k, p' \in \mathbf{Poly}$, an interaction pattern is a morphism of polynomials $p_1 \otimes \cdots \otimes p_k \to p'$, out of a Dirichlet product.

In a monoidal category, a monoid is an object p equipped with a unit and a multiplication operation satisfying certain laws. In the case of $\mathbf{Poly}$, the $\otimes$-monoids, which we call *collectives*, have surprisingly interesting semantics. A collective behaves as a sort of economic system that aggregates contributions and distributes returns accordingly. For example, one such collective is a prediction market where groups of analysts contribute their prediction of who will win some contest, and then the winner as well as some reward are distributed to each group, according to those odds (putting more odds on the actual winner results in getting higher returns). The associativity law ensures that economic systems such as this are scale-free.

We will search for more interesting examples, as well as various generalizations, e.g. categories enriched in $(\mathbf{Poly}, y, \otimes)$, to see what sort of semantically interesting and useful cases we can find.

Data-driven dynamics It was shown in [AU16] that comonoids in $(\mathbf{Poly}, y, \triangleleft)$ are precisely categories, and Garner has announced (and we've proven) that bimodules between them are parametric right adjoints, or as known in database circles, *data migration functors*. While we knew these facts at the time of the original grant application, we have begun to develop an understanding of how the database story and the dynamics story interact. It appears that coalgebras of these bimodules represent dynamical systems on agents who have a model of the world—an ontology—that includes possible ways things of one type can transform into things of another type. We aim to develop this intuition further and give examples. We also will consider how $(\otimes, \triangleleft)$-bimonoids correspond to a collective of agents operating together with the same ontology.

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